FLORIDA STATE UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

RIGIDITY OF POINT AND SPHERE CONFIGURATIONS:

AN EXAMINATION OF RIGIDITY IN LORENTZ, HYPERBOLIC, EUCLIDEAN, AND SPHERICAL GEOMETRY

By

OPAL GRAHAM

A Dissertation submitted to the
Department of Mathematics
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

2020

Opal Graham defended this dissertation on July 7, 2020.
The members of the supervisory committee were:

Philip L. Bowers
Professor Directing Dissertation

Nicholas Bonesteel
University Representative

Kathleen Petersen
Committee Member

Eric Klassen
Committee Member

The Graduate School has verified and approved the above-named committee members, and certifies that the dissertation has been approved in accordance with university requirements.

ACKNOWLEDGMENTS

I want to first thank Dr. Phil Bowers for being the best advisor I could have asked for, and for providing the right amount of guidance while encouraging independence, allowing me to grow as a mathematician. Phil and his collaborators, especially John Bowers and Ken Stephenson, have made my experience in Mathematics a wholly positive and inclusive one. I truly appreciate the time and opportunities Dr. Bowers and the Circle Packing group have afforded me.

I would like to extend thanks to the rest of my dissertation defense committee: Dr. Eric Klassen, Dr. Kate Petersen, and Dr. Nick Bonesteel for their time and effort. A special thanks to Dr. Petersen for her advice and help in applying for teaching positions. I also want to thank Dr. Ettore Aldrovandi for his continued kindness throughout my time in graduate school, which was especially vital in my first few years at FSU when I had a lot of catching up to do.

I have said it all along, but it bares repeating: I have come this far because of the support of the friends I made at FSU. Thank you to Chris Stover, Anindya Chanda, Dan Hartman, Thanittha Kowan, Haibin Hang, Zhe Su, and Sarah Algee for their friendship and help throughout the years. Thank you to Lorenzo Ruffoni and Alex Casella for the espressos and conversation. My gratitude especially goes to Michael Niemeier and Aamir Rasheed for sharing your insight, humor, and spare time with me.

I want to acknowledge my family and loved ones. I thank my parents for their endless support and for valuing education and learning. Thank you to my brothers and sisters, Nano, Chris, Tessi, and Alan for keeping me grounded and teaching me new things. Finally, I am so grateful to John Bergschneider for sharing this experience with me, for his help in difficult times, and for being a constant in my life.

CONTENTS

LIST OF FIGURES

ABSTRACT

This dissertation examines the study of rigidity of collections of objects in various geometric spaces, and the correspondences shared between geometries. In particular, we take a look at vectors and lines in Lorentz $(n+1)$-space, points, ideal points and hyperplanes in hyperbolic n-space, and circles and points in the Riemann sphere. One main objective is to explore how much information invariant to a given space is sufficient for a collection to be unique up to the transformations of the space. The answer to this question changes with the qualities a collection of objects possesses. To this end, this dissertation focuses on the role independence of objects plays in uniqueness. As another primary focus, a new invariant is introduced in each geometric setting to provide a means with which to study the rigidity of intermingled collections of objects that are infinitely far away from one another.

The first chapter gives a history of circle, sphere, and point configurations, and the correspondences between configurations of objects in hyperbolic space and Lorentz space. All theorems stated in this chapter are well-established results and provide both motivation for studying conditions for rigidity and for introducing an invariant that allows for intermingled collections of points and spheres.

In the second chapter, the main result is established entirely within the context of Lorentz space. An invariant of Lorentz transformations called the Lorentz ratio is defined that allows one to work with intermingled collections of vectors and light-like lines in Lorentz space. Three main statements are made about the rigidity of intermingled collections of vectors and light-like lines, each relying on a linearly independent collection of subspaces spanning the entire space. Each statement utilizes different information invariant to Lorentz space. These statements are crafted in Lorentz space with the intention of interpreting them in other geometric spaces.

In the third chapter, a dictionary between the objects and tools of Lorentz $(n+1)$-space and those in hyperbolic n-space is outlined so that the rigidity results in chapter 2 may be interpreted within the hyperbolic setting. Much of this information is standard and found within any given hyperbolic geometry text; some observations about the correspondence between linear independent vectors in Lorentz space and objects in hyperbolic space are novel. The Lorentz ratio also yields

an invariant in hyperbolic space we call the hyperbolic ratio. The main rigidity result of objects in hyperbolic space is stated at the end of the chapter.

In the final chapter, we turn our attention to the correspondence between objects in Lorentz space, and points and spheres in the $(n-1)$-sphere. There is an immediate correspondence that arises from the fact that the $(n-1)$-sphere can be taken as the ideal boundary of hyperbolic n-space. More specifically, circles and points in the Riemann sphere enjoy a geometry not dissimilar to points, lines, and planes in Euclidean space, so a notion of independence may be established within this geometry. This terminology is used to give a rigidity result for configurations of intermingled points and circles with independent collections of circles in the 2-sphere. Whereas inversive distance is a common conformal invariant used between pairs of circles, an inversive ratio between a point and two circles is defined so that these intermingled configurations may be considered. This chapter ends with results on the rigidity of inversive distance circle packings that use independence as a tool.

CHAPTER 1

INTRODUCTION AND HISTORICAL BACKGROUND

The rigidity of configurations of circles and points is important in several areas of complex analysis, geometry, and topology. As evidence, one need only refer to work from Paul Koebe, E.M. Andre'ev, and William Thurston. All three mathematicians have contributions to the study of *circle packings* (circle patterns on a surface with an underlying triangulation specifying circle overlaps) in the form of the Koebe-Andre'ev-Thurston (KAT) theorem [4]. A special case of this theorem, where all overlaps are tangencies, is stated here.

Koebe Circle Packing Theorem ([16]). *Given a triangulation $\mathcal{K}$ of a topological sphere, there exists a tangency circle packing $\mathcal{K}(\mathcal{C})$ on the Riemann sphere $\mathbb{S}^2$ with the combinatorics of $\mathcal{K}$. The circle packing $\mathcal{K}(\mathcal{C})$ is unique up to Möbius transformations of the sphere.*

This theorem is the work of Koebe in 1936. Thurston generalized this theorem to the statement of the KAT theorem, which involves circle packings that allow for given overlaps, and allows for triangulations of arbitrary compact orientable surfaces. In [18], Thurston states the Koebe Circle Packing Theorem without proof, attributing it to Andre'ev rather than Koebe. Thurston was not aware of the theorem by Koebe, but noted that his theorem for circles translated to a characterization of three-dimensional convex hyperbolic polyhedra, which Andre'ev accomplished in 1970. This correspondence is seen easily when using the Klein Model of hyperbolic 3-space: in this model, $\mathbb{S}^2$ serves as the ideal boundary of $\mathbb{H}^3$, and each face of a hyperbolic polyhedron is supported by a Euclidean plane intersecting $\mathbb{S}^2$. The collection of supporting Euclidean planes intersect $\mathbb{S}^2$ as a collection of circles. Thurston generalized this statement to the KAT theorem in order to build hyperbolic structures on orbifolds.

Topologists continue to use circle packings and the KAT theorem for building hyperbolic structures on manifolds. The KAT theorem enjoys many other uses, including Thurston's demonstration that Koebe's result may be used to effectively approximate the Riemann mapping from a proper

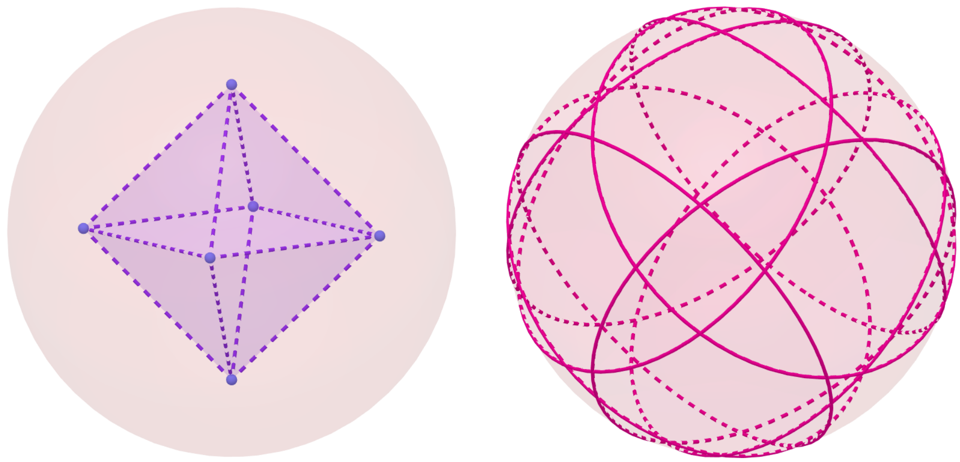

Figure 1.1: A hyperbolic polyhedron (left), and its corresponding circle packing in $\mathbb{S}^2$ (right).

simply-connected, planar domain to the unit disk. In this way, circle packings are used in building *conformal tilings* – tilings of surfaces with specified angle patterns. These tilings have been used by a number of mathematicians in their effort to solve the Cannon Conjecture [10], [8], [9], [11]. For a more complete portrayal of the design and consequences of the KAT theorem, see [4].

Koebe pioneered many other statements crucial to the growth of discrete conformal geometry in his efforts to solve his own famed uniformization conjecture.

Koebe Uniformization Conjecture ([14])**.** *Every domain in the Riemann sphere is conformally homeomorphic to a circle domain.*

The term *circle domain* refers to a connected open set with complementary components, all of which are points or closed round disks. In 1920, he showed in [15] that every finitely-connected domain in the Riemann sphere is conformally equivalent to a circle domain, where *finitely-connected* refers to finitely many boundary components of the domain in $\mathbb{S}^2$. This is a generalization of the Riemann Mapping Theorem, to which Koebe's conjecture reduces when the domain is 1-connected. He also proved the rigidity statement that any conformal homeomorphism between two circle domains with finitely many complementary components is a restriction of a Möbius transformation [12].

Beardon and Minda, cite Koebe's rigidity statement as a fundamental piece of machinery for the proof of their main theorem in [3]. They prove a statement for *circular regions* in the extended

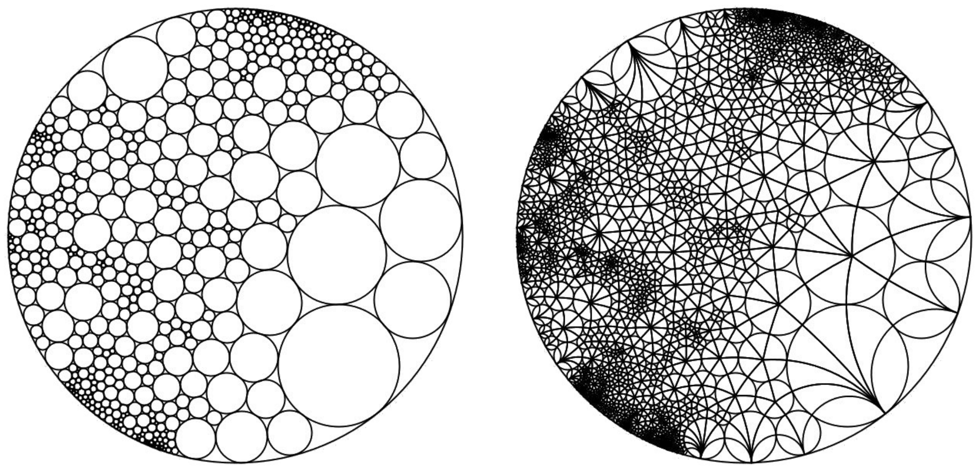

Figure 1.2: A tangency circle packing of the hyperbolic plane (left), and its underlying triangulation (right). Each vertex in the triangulation represents a circle. Each edge between vertices means the two circles are tangent.

complex plane $\hat{\mathbb{C}}$, regions bounded by a collection of pairwise disjoint circles, using the conformal invariant *inversive distance*, a real number measuring the separation between a pair of circles.

Definition 1.0.1. *Let C_1 and C_2 be oriented circles in $\mathbb{S}^2$. When C_1 and C_2 are intersecting, the* ***inversive distance between C_1 and C_2****, denoted (C_1, C_2) is*

$$(C_1, C_2) = \cos \alpha,$$

where α is the oriented angle of intersection of C_1 and C_2. When C_1 and C_2 are disjoint,

$$(C_1, C_2) = \cosh d_{\mathbb{H}^2}(\ell_1, \ell_2),$$

where $\ell_1 = D \cap C_1$, and $\ell_2 = D \cap C_2$, for a disc D mutually orthogonal to C_1 and C_2, used as a model of the hyperbolic plane. As such, $d_{\mathbb{H}^2}(\ell_1, \ell_2)$ is the hyperbolic distance between ℓ_1 and ℓ_2.

When the circles are unoriented, the inversive distance is the absolute value of the inversive distance between the two circles when each is given either orientation; when the circles are oriented, meaning an interior disk is chosen to accompany each, this choice may yield a positive or negative inversive distance. Various equivalent formulas for inversive distance are discussed at length in the last chapter.

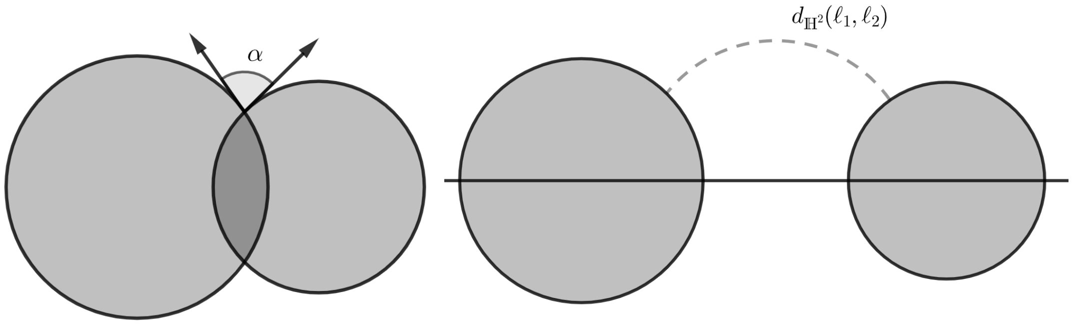

Figure 1.3: When oriented circles C, C' are intersecting, $-1 \leq \text{InvDist}(C, C') \leq 1$ (left). When C, C' are disjoint, $1 < \text{InvDist}(C, C') < \infty$, or $\infty < \text{InvDist}(C, C') < -1$ (right).

Theorem 1.0.2 (Beardon an Minda)**.** *Suppose that* Ω *and* Ω' *are circular regions bounded by circles* $C_1, \ldots, C_m$ *and* $C'_1, \ldots, C'_m$*, respectively, where* $m \geq 2$*. There is a Möbius transformation* f *with* $f(\Omega) = \Omega'$ *and* $f(C_j) = C'_j$*,* $1 \leq j \leq m$*, if and only if* $(C_j, C_k) = (C'_j, C'_k)$ *for all* j *and* k *with* $1 \leq j < k \leq m$*.*

They also make the following rigidity statement for m-punctured spheres, via collections of m points on the sphere. Here, Beardon and Minda make use of a conformal invariant of ordered 4-tuples of points, the *absolute cross ratio*, denoted $|a, b, c, d|$ for points a, b, c, d in the Riemann sphere.

Theorem 1.0.3 (Beardon and Minda)**.** *Given two collections of points* $p_1, \ldots, p_m$ *and* $p'_1, \ldots, p'_m$ *in* $\hat{\mathbb{C}}$*,* $m \geq 4$*, there is a Möbius transformation* f *with* $f(p_i) = p'_i$ *for* $i = 1, 2, \ldots, m$ *if and only if* $|p_i, p_j, p_k, p_l| = |p'_i, p'_j, p'_k, p'_l|$ *for all distinct* $1 \leq i, j, k, l \leq m$*.*

Beardon and Minda pose a series of questions at the end of [3]; namely, they ask whether their first result can be extended by including circles which intersect, and whether both statements generalize to higher dimensions. In [13], Crane and Short answer both questions in the affirmative, provided reasonable conditions are met. They are also able to extend each statement to collections of uncountably many spheres and uncountably many points. In [13], Crane and Short pose the statement in terms of collections of balls, citing that to each sphere, one can assign an interior ball. We do the same, using the language of oriented spheres, denoted C_α, and refer to the same sphere with opposite orientation as $\overline{C_\alpha}$. The two statements are as follows.

Theorem 1.0.4 (Crane and Short)**.** *Let* $\{C_\alpha : \alpha \in \mathcal{A}\}$ *and* $\{C'_\alpha : \alpha \in \mathcal{A}\}$ *be two collections of oriented spheres in* $\hat{\mathbb{R}}^{n+1}$*, indexed by the same set. Suppose that* $\cap_{\alpha\in\mathcal{A}} C_\alpha = \emptyset$*. Then there is a Möbius transformation* f *such that one of the following holds: either* $f(C_\alpha) = C'_\alpha$ *for each* α *in* $\mathcal{A}$*, or else* $f(\overline{C_\alpha}) = C'_\alpha$ *for each* α *in* $\mathcal{A}$*, if and only if* $(C_\alpha, C_\beta) = (C'_\alpha, C'_\beta)$ *for all pairs of* α *and* β *in* $\mathcal{A}$*.*

Theorem 1.0.5 (Crane and Short)**.** *Let* $\{p_\alpha : \alpha \in \mathcal{A}\}$ *and* $\{p'_\alpha : \alpha \in \mathcal{A}\}$ *be two collections of distinct points in* $\hat{\mathbb{R}}^n$*, indexed by the same set. There is a Möbius transformation* f *with* $f(p_\alpha) = p'_\alpha$ *for each* α *in* $\mathcal{A}$ *if and only if* $|p_\alpha, p_\beta, p_\gamma, p_\delta| = |p'_\alpha, p'_\beta, p'_\gamma, p'_\delta|$ *for all ordered* 4*-tuples* $(\alpha, \beta, \gamma, \delta)$ *of distinct indices in* $\mathcal{A}$*.*

Both Beardon and Minda, and Crane and Short use the term *Möbius transformation* to refer to either a conformal or anti-conformal map. For our purposes, we will make the distinction between the two by referring to a map g of $\hat{\mathbb{C}}$ where $g(z) = (az + b)/(cz + d)$ and $ad - bc \neq 0$ as a ***Möbius transformation*** of $\hat{\mathbb{C}}$, while a map h where h may be either conformal or anti-conformal, $h(z) = g(\overline{z})$ or $h(z) = g(z)$, will be referred to as an ***inversive transformation*** of $\hat{\mathbb{C}}$.

While the statements made by Beardon and Minda are reminiscent of Koebe's work with circle domains, there are a few differences. One is Beardon and Minda's involvement of inversive geometry in their statements, via the introduction of absolute cross ratio and inversive distance. The other is that both Beardon and Minda, and Crane and Short, make two separate rigidity statements: one for circles (spheres), and one for points. The natural question arises: Can the work of Crane and Short (and by extension, Beardon and Minda), be generalized even further to a rigidity statement, up to inversive transformation, of *intermingled* collections of spheres and points? This dissertation demonstrates that this can be accomplished, and with markedly less conformal invariant information used than in any of the four rigidity statements above. The obvious issue is that inversive distance only uses circles (and spheres) as input, and the absolute cross ratio only uses points. This matter is overcome by introducing a new conformal invariant, referred to as the *inversive ratio* of a point and two spheres. The concept behind the inversive ratio is to view a point p in $\mathbb{S}^{n-1}$ as the limit of a sequence C_j of spheres, where, as $r_j \to 0$, $C_j \to p$. Take two fixed spheres C and C' in $\mathbb{S}^{n-1}$ not in the sequence. Then the two inversive distances (C_j, C) and (C_j, C') grow unbounded at the same rate as $r_j \to 0$. Provided p does not lie in C or C', the ratio of these two sequences of inversive distances limits to the inversive ratio, a real number, denoted (p, C, C').

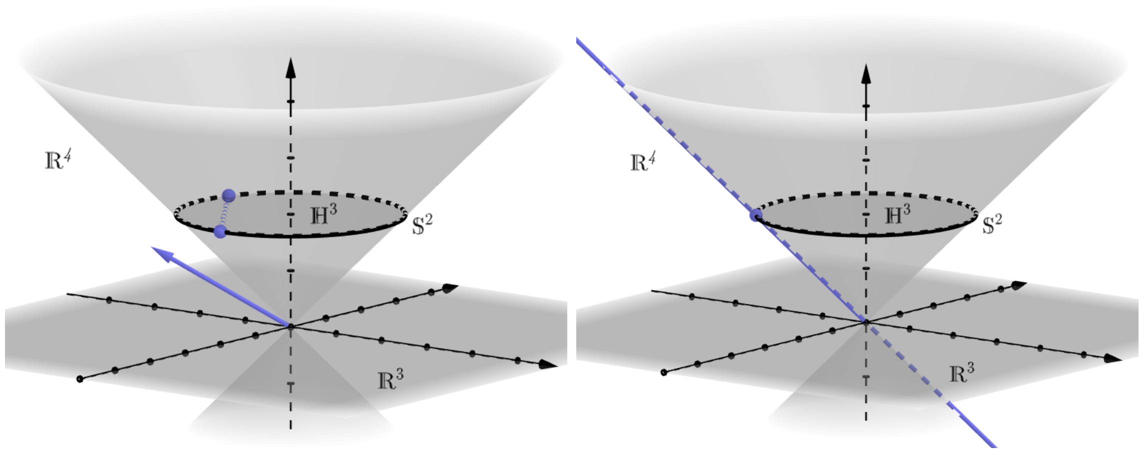

Figure 1.4: The *light cone* is a prominent feature of Lorentz space. Space-like vectors lie outside the light cone, time-like vectors lie inside the light cone, and light-like vectors lie on the light cone. For every space-like vector v, there is a time-like n-dimensional subspace V such that for all $w \in V$, $\langle v, w \rangle = 0$, where V intersects hyperbolic n-space in a hyperplane P, and intersects $\mathbb{S}^{n-1}$ in a sphere C acting as the ideal boundary of P. Light-like lines are Lorentz orthogonal to n-dimensional light-like subspaces, intersecting the ideal boundary of hyperbolic n-space as an ideal point. This is the basis of the correspondence used between chapters.

In this dissertation, one objective is to state a rigidity theorem for intermingled points and spheres; this statement is made in chapter 4. Chapters 2 and 3 develop the machinery that ultimately yields this statement. Crane and Short do the work for proving their rigidity statements in the context of Lorentz space, $\mathbb{R}^{n+1}$ equipped with the Lorentz inner product. Here, the Lorentz inner product of vectors $v, w \in \mathbb{R}^{n+1}$ is denoted $\langle v, w \rangle$, and separates vectors in Lorentz space into three different types, *space-like, time-like, and light-like.* In chapter 2, we follow the trend set by Crane and Short and consider rigidity of vectors and light-like lines in Lorentz space, independent of the geometric meaning in other settings. The Lorentz inner product takes vectors as input, but for the purposes of this dissertation, a measure is needed between space-like or time-like vectors and light-like lines. With this in mind, an invariant of Lorentz transformations, called the *Lorentz ratio* of a light-like line and two fixed vectors, is given; this and a basis of vectors is used in the main rigidity result of the chapter.

In chapter 3, we begin exploring the geometric meaning of rigidity statements made in Lorentz space. When v and w are space-like unit vectors, they correspond to hyperplanes P_v, P_w in hyperbolic n-space, and $\langle v, w \rangle$ corresponds to the hyperbolic distance between P_v and P_w. Likewise, when v and w are positive time-like unit vectors, they correspond to points in hyperbolic n-space, and $\langle v, w \rangle$ corresponds to hyperbolic distance between the hyperbolic points. A point a in the ideal boundary of hyperbolic n-space is compared with hyperbolic points of hyperplanes via the correspondence between the Lorentz ratio and *hyperbolic ratio*. Hyperbolic isometries are restrictions of positive Lorentz transformations; the geometry of hyperbolic space and its ideal boundary are extrinsically encoded in the geometry of Lorentz space. It is within hyperbolic space that the most general translation of our rigidity result for vectors and light-like lines can be realized as a rigidity statement of points, ideal points, and hyperplanes of hyperbolic n-space.

In Chapter 4, the rigidity statements in Chapter 2 are restated in the language of inversive geometry. Special attention is given to the case in which the dimension of Lorentz Space is $n + 1 = 4$. Here, a rigidity statement is made for intermingled points and circles in $\mathbb{S}^2$, where the collections contain an *independent subcollection of* 4 *circles*. In Euclidean space, a line can be uniquely determined by two distinct points; a plane is determined by three linearly independent points. In the geometry of circles, a circle line (coaxial family) can be determined by two distinct circles. Without going into detail here, a circle-plane is determined by three *independent* circles in $\mathbb{S}^2$, meaning three circles which do not belong to a common coaxial family of circles. Four circles are independent if they do not all belong to a common circle-plane. Below, a corollary to one of the main theorem's is stated. Note that since Ω and Ω' are circle domains, all oriented circles are disjoint. The main rigidity theorem allows for circles with non-trivial intersection.

Theorem 1.0.6. *Let Ω and Ω' be two circle domains, respectively bounded by collections of oriented circles and points, $\{C_\alpha, p_\beta : \alpha, \beta \in \mathcal{A}\}$ and $\{C'_\alpha, p'_\beta : \alpha, \beta \in \mathcal{A}\}$, in $\mathbb{S}^2$. Suppose each collection has an independent subcollection of 4 circles, $\{C_1, C_2, C_3, C_4\}$ and $\{C'_1, C'_2, C'_3, C'_4\}$ respectively, where $(C_i, C_j) = (C'_i, C'_j)$ for each distinct pair $1 \leq i, j \leq 4$. Then there is an inversive transformation ϕ such that one of the following holds: either $\phi(C_\alpha) = C'_\alpha$ and $\phi(p_\beta) = p'_\beta$ for each α, β in $\mathcal{A}$ or else $\phi(\overline{C_\alpha}) = C'_\alpha$ and $\phi(p_\beta) = p'_\beta$ for each α, β in $\mathcal{A}$, if and only if $(C_\alpha, C_i) = (C'_\alpha, C'_i)$ and $(p_\beta, C_i, C_j) = (p'_\beta, C'_i, C'_j)$ for all distinct α, β, i, j.*

The other facet considered by this dissertation, in chapter 4, is whether the combinatorics of the configurations sufficient for rigidity. As stated above, the results of Beardon and Minda and Crane and Short, use a maximal amount of inversive distance and absolute cross ratio information,

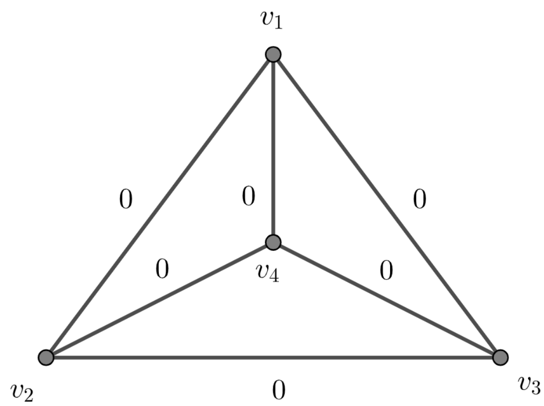

Figure 1.5: A tetrahedral triangulation of $\mathbb{S}^2$, where all edge-labels are 0. There is no collection of circles which realizes this triangulation, so no such circle-packing exists.

whereas the main results of this dissertation use less. This is achieved by requiring an additional (but reasonable) condition that the collections have maximally independent subcollections. While the addition is reasonable, in Chapter 4, we end by turning our attention back to more commonly used configurations of circles. In particular, we look at generalizations of circle packings, called *inversive distance circle packings (IDCPs)*. Here, adjacent circles may intersect at an angle, be tangent, or disjoint. The edges in the corresponding triangulation are equipped with a real number specifying the inversive distance.

Broadening the view to IDCPs creates difficulties: for one, the KAT theorem doesn't generalize to IDCPs. Not all edge-labeled triangulations have an IDCP realization. Seeing this is as simple as taking a tetrahedral graph and labeling all edges with 0, depicted above. A collection of circles $\{C_1, C_2, C_3, C_4\}$, with each C_i corresponding to vertex v_i in the triangulation, would need to be such that C_1 and C_2 determine a hyperbolic coaxial family of circles, $\mathcal{A}_{C_1,C_2}$ and circle C_4 orthogonal to both circles must necessarily be in the unique elliptic coaxial family of circles orthogonal to $\mathcal{A}_{C_1,C_2}$. The same is true of C_3, but all circles in an elliptic coaxial family are disjoint, so there is no such collection of circles realizing this edge-labeled triangulation.

Additionally, not all IDCPs are globally unique up to Möbius transformation; in [6], an example is constructed with an octahedral graph triangulating $\mathbb{S}^2$, edge-labeling as assigned in figure 1.6. A

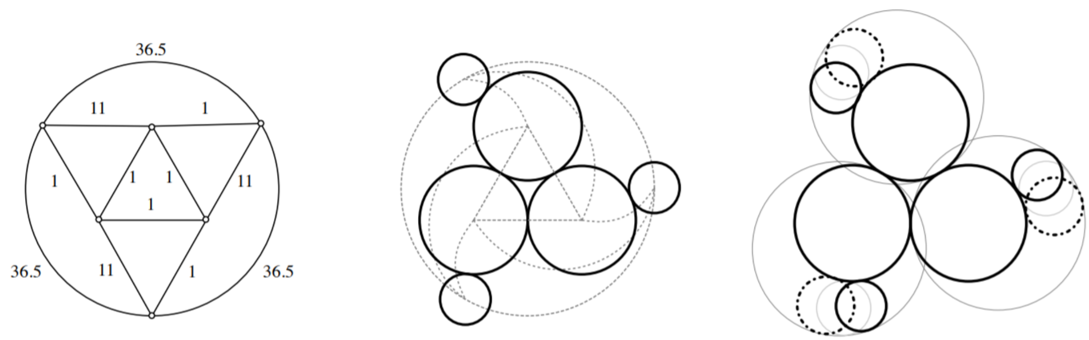

Figure 1.6: Image from [6]. An octahedral graph, edge-labeled with inversive distances (left); a planar circle pattern realzing the edge-labeled octahedral graph (center); Two circle patterns non-Möbius-equivalent with inversive distance 37 on outer edges.

circle packing realizing this inversive distance pattern is called a *critical packing*, where the outer circles are a minimal inversive distance from one another. There are two circle realizations of this octahedral graph with no Möbius transformation between the two collections: the inner circles remain fixed while the outer circles are rotated. IDCPs are special cases of circle configurations with underlying edge-labeled polyhedral graphs (3-vertex-connected, planar graphs) — such configurations are called *circle-polyhedra*, or *c-polyhedra* for short. Under this lens, powerful tools from the rigidity theory of polyhedra may be reworked to address the uniqueness of IDCPs; the famous Cauchy's Rigidity Theorem is one such tool. Cauchy's Rigidity Theorem states that two convex, combinatorially equivalent, bounded Euclidean polyhedra with corresponding congruent faces are themselves congruent. Convexity is a powerful notion in the discipline of polyhedral geometry (see [17]), so it is a natural step to introduce an analogous notion for circle polyhedra.

Circles have their own geometry under Möbius transformations acting on $\mathbb{S}^2$, where there is a notion of a *circle point* (a circle in $\mathbb{S}^2$), *circle line*, and *circle plane*. There is a strong connection between circle polyhedra in $\mathbb{S}^2$ and *projective polyhedra*. If we consider $\mathbb{RP}^3 = \mathbb{E}^3 \cup \mathbb{RP}^2$ as our model of real projective space, a projective polyhedron can always be transformed to look like a bounded Euclidean polyhedron. Within $\mathbb{E}^3$, the unit sphere $\mathbb{S}^2$ serves as the ideal boundary for the Klein model of hyperbolic space $\mathbb{H}^3$. Several cases of projective polyhedra have been classified. In [21] Andre'ev classified hyperbolic convex polyhedra with acute dihedral angles. In [20] Rivin classified hyperbolic convex polyhedra with vertices at the ideal boundary, called *ideal polyhedra*. Bao and Bonahon in [1] push those vertices past the ideal boundary to classify *hyperideal polyhedra*,

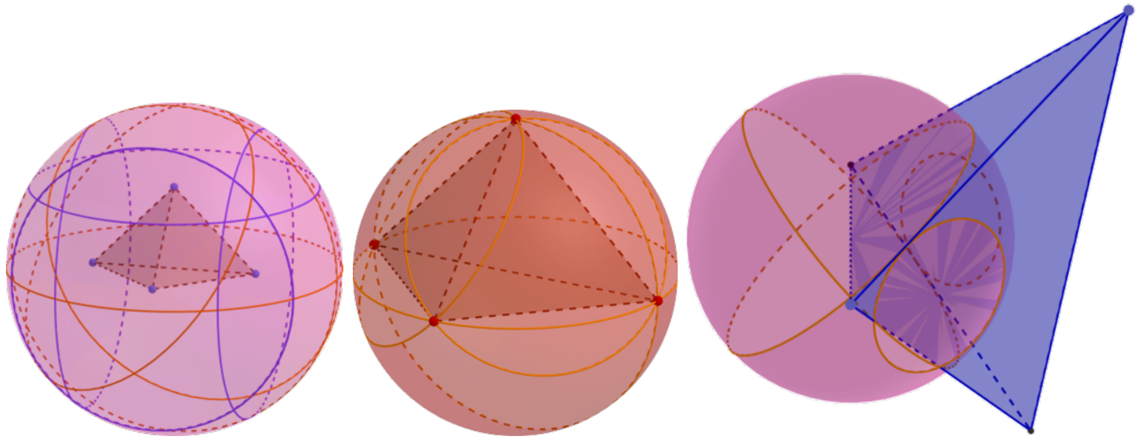

Figure 1.7: Projective polyhedra and their corresponding circle polyhedra. Left is a hyperbolic polyhedron lying entirely within hyperbolic space, center is an ideal polyhedron where vertices lie on the ideal boundary, and right is a hyperideal polyhedron, where vertices push past the ideal boundary.

where every edge must intersect hyperbolic space. One primary reason topologists are interested in ideal polyhedra is because of their use as building blocks when constructing hyperbolic 3-manifolds [1].

Most recently, Bowers, Bowers, and Pratt in [5] explored rigidity of more general hyperideal polyhedra. Their result is stated in terms of corresponding circle polyhedra in the 2-sphere.

Theorem 1.0.7 (BBP)**.** *Any two convex and proper non-unitary circle polyhedra with Möbius-congruent circle faces that are based on the same oriented abstract spherical polyhedron and are consistently oriented are Möbius-congruent.*

Bowers, Bowers, and Pratt introduced a notion of convexity analogous to the definition for Euclidean polyhedra, in that all circle points must ”lie on one side” of each circle face, bounded by a certain inversive distance. This result directly implies that all convex IDCPs are rigid.

In general, like Euclidean polyhedra, circle-polyhedra, and IDCPs, are not convex. Rather, what we do have with general c-polyhedra is an assumption that the c-faces are non-degenerate (the circles realizing the vertices of a face in the polyhedral graph do not all lie in a coaxial family of circles). With IDCPs, this affords the assumption that any four circles realizing two adjacent faces in the triangulation must be independent. With this information, the last part of chapter 4 concerns answering the question “how much extra inversive distance information is needed to make a general inversive distance circle packing of $\mathbb{D}$ and of $\mathbb{S}^2$ rigid?" Extra inversive distance information is assumed by adding edges across edges of adjacent faces in a triangulation of $\mathbb{D}$. An algorithm for adding in enough edges to make two collections of oriented circles originating from a circle packing is described.

CHAPTER 2

VECTORS AND LINES IN LORENTZ SPACE

One of the main goals of this paper is to find a uniqueness statement for intermingled points and $(n-2)$-spheres in $\mathbb{S}^{n-1}$, up to Möbius transformations. Analyzing these geometric objects together is a natural consequence of considering the ambient space, $\mathbb{R}^{n+1}$, equipped with a non-degenerate symmetric bilinear form, that $\mathbb{S}^{n-1}$ lies within. Configurations of intermingled $(n-2)$-spheres and points in the $(n-1)$-sphere, and hyperbolic points in $\mathbb{H}^n$, correspond to configurations of vectors and lines in Lorentz Space, and Möbius transformations correspond to Lorentz transformations. Studying these configurations in Lorentz Space allows one to determine rigidity by taking advantage of the ease of linear algebra computations.

In this chapter, the geometry of vectors and lines in Lorentz n-space is the focus, untethered to its relationship with the geometry of circles. The basics of Lorentz Space and Lorentz transformations are introduced. Then, the rigidity of configurations of vectors and lines is explored using various Lorentz invariants. Rigidity of intermingled vectors and lines is achieved by introducing a new Lorentz invariant. In the chapter after, the correspondence between points and spheres in $\mathbb{S}^{n-1}$ and vectors and lines in Lorentz Space is fully detailed, so that rigidity of intermingled points and spheres is attained.

2.1 Lorentz Space Basic Definitions and Propositions

This section contains definitions and propositions regarding Lorentz Space that will either be relevant in this chapter or later in connecting the extrinsic geometry of Lorentz Space to the intrinsic geometry of Circle Space. In this chapter, all objects are considered within an $(n+1)$-dimensional setting, $\mathbb{R}^{n+1}$. The precise vector space is defined below.

Definition 2.1.1. *Let v, w be two vectors in $\mathbb{R}^{n+1}$. The* ***Lorentz inner product*** *$\langle v, w\rangle$ between v and w is*

$$\langle v, w\rangle = v_1w_1 + v_2w_2 + ... + v_nw_n - v_{n+1}w_{n+1}. \tag{2.1}$$

$\mathbb{R}^{n+1}$ equipped with the Lorentz inner product is called ***Lorentz space.***

Note that the Lorentz inner product is not an actual inner product: in particular, it is not positive-definite. For example, take vectors $v = (1, 2, 1, 5)$ and $w = (1, 1, 1, 1)$ in $\mathbb{R}^4$ and observe that $\langle v, w\rangle = -1$. However, it is a symmetric bilinear form, and it satisfies the weaker condition of being non-degenerate. The following definition of non-degeneracy is only true in finite-dimensional vector spaces.

Definition 2.1.2. *Let V be a finite-dimensional vector space. Let $B(\cdot, \cdot)$ denote a bilinear form on on V. Then $B(\cdot, \cdot)$ is* ***non-degenerate*** *if $v = 0$ whenever $B(v, w) = 0$ for all w in V.*

For finite-dimensional vector spaces, $B(\cdot, \cdot)$ is non-degenerate exactly when $\det [B]_F \neq 0$, where $[B]_F$ is the matrix associated with $B(\cdot, \cdot)$ relative to a basis F.

Let $F = \{f_1, \dots, f_{n+1}\}$ be any basis for $\mathbb{R}^{n+1}$. Let $\{e_i\}$ denote the standard basis for $\mathbb{R}^{n+1}$, and let ω be the matrix representing the change of basis from $\{f_i\}$ to $\{e_i\}$. For Lorentz inner product $\langle \cdot, \cdot\rangle$, the associated matrix is

$$\langle \cdot, \cdot\rangle = [B] = \left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & \ddots & & \vdots & \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & \\ \hline & & 0 & & -1 \end{array}\right) = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & -1 \end{array}\right). \tag{2.2}$$

Note that $\det([B]) \neq 0$. For basis F, the matrix associated with the bilinear form $B_F(\cdot, \cdot)$ is $B_F(\cdot, \cdot) = [B]_F = \omega[B]\omega^t$, where ω^t is the transpose of ω. Then $\det([B]_F) = \det(\omega[B]\omega^t) = \det(\omega)\det([B])\det(\omega^t) = \det(\omega)^2 \det([B]) \neq 0$.

Lemma 2.1.3. *Let $\{f_1, \dots, f_n, f_{n+1}\}$ be a basis for $\mathbb{R}^{n+1}$. Let v be a vector in $\mathbb{R}^{n+1}$ such that $v \neq f_i$ for each $i = 1, \dots, n + 1$. If $\langle v, f_i\rangle = 0$ for every i, then $v = 0$.*

Proof. Let $F = \{f_1, \dots, f_n, f_{n+1}\}$ be a basis for $\mathbb{R}^{n+1}$. Since $\langle \cdot, \cdot\rangle$ is non-degenerate, by definition, if $\langle v, w\rangle = 0$ for every $w \in \mathbb{R}^{n+1}$, then $v = 0$. Since $\langle v, f_i\rangle = 0$ for every $i = 1, \dots, n + 1$, then for any $w = b_1 f_1 + ... + b_n f_n + b_{n+1} f_{n+1}$, we obtain $\langle v, w\rangle = \langle v, b_1 f_1 + \dots + b_{n+1} f_{n+1}\rangle = b_1\langle v, f_1\rangle + \dots + b_{n+1}\langle v, f_{n+1}\rangle = 0$ and so $v = 0$. $\square$

The non-degeneracy of the Lorentz inner product is a crucial factor in the arguments for the proceeding rigidity statements concerning vectors and lines. Vectors (and, in general, subspaces) in Lorentz space come in three flavors: space-like, time-like, or light-like. This categorization also plays a key role in how vectors will interact with one another within the Lorentz inner product.

We will refer to $\langle v, v\rangle = ||v||^2$ as the **Lorentz norm of** v, while the Euclidean norm will be denoted $v \cdot v = |v|^2$.

Definition 2.1.4. *A vector is* ***space-like*** *if* $||v||^2 > 0$*,* ***light-like*** *if* $||v||^2 = 0$*, or* ***time-like*** *if* $||v||^2 < 0$*. Let* V *denote a subspace of* $\mathbb{R}^{n+1}$*. Subspace* V *is said to be* ***time-like*** *if there is a time-like* v *in* V*,* ***space-like*** *if every non-zero vector is space-like, and* ***light-like*** *otherwise. The set of all* v *such that* $||v||^2 = 0$ *is called the* ***light cone****, denoted* C^n*.*

Definition 2.1.5. *Let* v *be a space-like or time-like vector. Vector* v *is a* ***unit vector*** *if* $||v||^2 = 1$*, or respectively, if* $||v||^2 = -1$*.*

There is no notion of a "light-like unit vector" because $||v||^2 = 0$ whenever v is light-like.

Definition 2.1.6. *Two vectors* v *and* w *in* $\mathbb{R}^{n+1}$ *are* ***Lorentz orthogonal*** *if* $\langle v, w\rangle = 0$*.*

Definition 2.1.7. *Two vectors* v, w *in* $\mathbb{R}^{n+1}$ *are* ***Lorentz orthonormal*** *if and only if* $||v||^2 = -1$ *and* $\langle v, w\rangle = 0$ *and* $||w||^2 = 1$*.*

Definition 2.1.8. *Let* V *be a subspace of* $\mathbb{R}^{n+1}$*. The subspace* $V^L = \{w \in \mathbb{R}^{n+1} : \langle w, v\rangle = 0, \forall v \in V\}$ *is the* ***Lorentz complement*** *of* V*.*

Unlike the Euclidean scalar product, $\langle v, w\rangle = 0$ does not mean that v and w are perpendicular, necessarily. Indeed, the Lorentz norm of any light-like vector confirms this. As another example, take $v = (2, 0, 0, 1)$ and $w = (1, 0, 0, 2)$ in $\mathbb{R}^4$. Then $\langle v, w\rangle = 0$, but $v \cdot w = 4$, with $|v||w| = 5$, so $\cos\theta = 4/5$. We will see that a similar relationship will be set up between vectors. Consider the following lemmas, stated in [19].

Lemma 2.1.9 ([19])**.** *Let* v *and* w *be two nonzero vectors in* $\mathbb{R}^{n+1}$ *which are Lorentz orthogonal. If* v *is time-like, then* w *is space-like.*

Lemma 2.1.10 ([19])**.** *The subspace* V *is time-like in* $\mathbb{R}^{n+1}$ *if and only if* V^L *is space-like.*

Since Lorentz space is a vector space, any subspace V such that $\dim V = m$ has a Lorentz complement V^L such that $\dim V^L = n + 1 - m$, and further, $(V^L)^L = V$. This means Lemma 2.1.10 can rephrased to say that Subspace V is space-like if and only if V^L is time-like. Moreover, this leads to the following corollary.

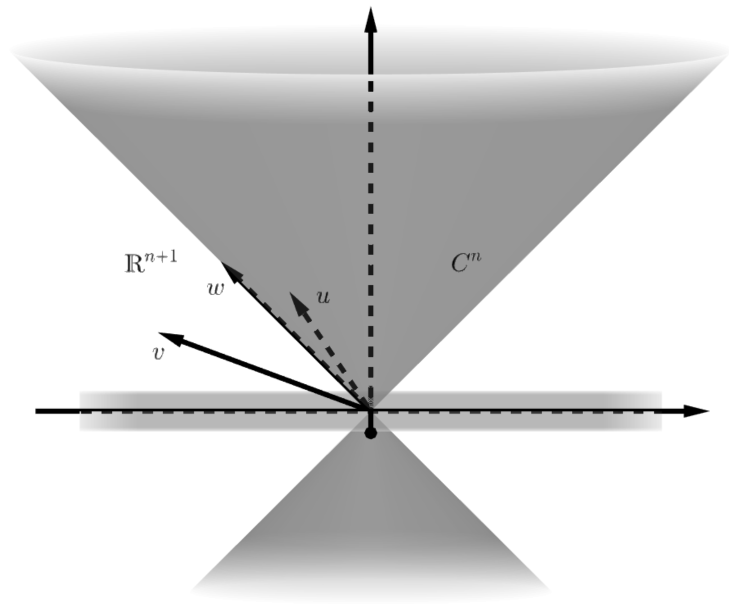

Figure 2.1: The light cone C^n is a prominent feature of Lorentz space. Space-like vectors lie outside the light cone, time-like vectors lie inside the light cone, and light-like vectors lie on the light cone. More generally, a subspace is space-like if it does not intersect the light cone, light-like if it intersects the light cone in a line, and time-like otherwise. Depicted, v is space-like, w is light-like, and u is time-like.

Corollary 2.1.11. *The subspace V is light-like if and only if V^L is light-like, where $V \cap C^n = V^L \cap C^n = \ell$, where ℓ is a line through the origin.*

One should note that the converse statement of Lemma 2.1.9 is not true. If v and w are Lorentz orthogonal, and v is space-like, w may be time-like, space-like, or light-like. As an example, take $v = \langle 2, 0, 0, 0 \rangle$ and $w_1 = \langle 0, 3, 0, 0 \rangle$ in $\mathbb{R}^4$. Observe that both vectors are space-like because $||v||^2 = 4$ and $||w_1||^2 = 9$. However, $\langle v, w_1 \rangle = 0$. For time-like $w_2 = \langle 1, 0, 0, -5 \rangle$, observe that $\langle v, w_2 \rangle = 0$. Finally, for light-like $w_3 = \langle 0, 2, 0, 2 \rangle$, note that $\langle v, w_3 \rangle = 0$.

Definition 2.1.12. *A vector v in $\mathbb{R}^{n+1}$ is **positive** if $v_{n+1} > 0$, and **negative** if $v_{n+1} < 0$.*

The following statements from [19] are statements analogous to the Euclidean scalar product statement $v \cdot w = |v||w| \cos \theta$.

Theorem 2.1.13 ([19])**.** *Let v, w be positive (negative) time-like vectors in $\mathbb{R}^{n+1}$. Then*

$$\langle v, w \rangle \leq ||v||||w||, \tag{2.3}$$

with equality if and only if v and w are linearly dependent.

Corollary 2.1.14 ([19]). *For positive (negative) time-like vectors v and w in $\mathbb{R}^{n+1}$, there is a unique nonnegative real number $\eta(v, w)$ such that*

$$\langle v, w\rangle = ||v||||w|| \cosh \eta(v, w). \tag{2.4}$$

Definition 2.1.15. *The nonnegative real number $\eta(v, w)$ is called the **Lorentz time-like angle** between v and w.*

Note for any two time-like unit vectors v, w that theorem 2.1.13 implies

$$\langle v, w\rangle \leq ||v||||w|| = (\sqrt{-1})(\sqrt{-1}) = -1.$$

Space-like vectors follow a similar pattern, but there are more cases to consider.

Theorem 2.1.16 ([19]). *Let v and w be linearly independent space-like vectors in $\mathbb{R}^{n+1}$ Then,*

1. *v and w satisfy the inequality $|\langle v, w\rangle| < ||v||||w||$ if and only if the subspace V spanned by v and w is space-like;*
2. *v and w satisfy the inequality $|\langle v, w\rangle| > ||v||||w||$ if and only if the subspace v spanned by v and w is time-like.*
3. *v and w satisfy the equation $|\langle v, w\rangle| = ||v||||w||$ if and only if the subspace spanned by v and w is light-like.*

In the case of 1, there is a unique real number $0 < \eta(v, w) < \pi$ such that

$$\langle v, w\rangle = ||v||||w|| \cos \eta(v, w). \tag{2.5}$$

This equation holds for 3 when $\eta(v, w)$ is 0 or π. Note that in this case, $\eta(v, w) = 0$ when v and w are positive scalar multiples of one another, and $\eta(v, w) = \pi$ when v and w are negative scalar multiples.

Definition 2.1.17. *In (2.5), the unique real number $\eta(v, w)$ is called the **Lorentz space-like angle** between v and w.*

In case 2, there is a unique positive real number $\eta(v, w)$ such that

$$|\langle v, w\rangle| = ||v||||w|| \cosh \eta(v, w). \tag{2.6}$$

Definition 2.1.18. *The unique real number* $\eta(v, w)$ *in (2.6) is called the* ***Lorentz time-like angle*** *between* v *and* w*.*

Observe that in the case where v and w are space-like unit vectors spanning a time-like subspace V,

$$|\langle v, w\rangle| > ||v||||w|| = 1, \tag{2.7}$$

and when v and w span a space-like subspace,

$$0 < |\langle v, w\rangle| < 1. \tag{2.8}$$

Theorem 2.1.19 ([19])**.** *Let* v *be a space-like vector and* w *a positive time-like vector. Then there is a unique nonnegative real number* $\eta(v, w)$ *such that*

$$|\langle v, w\rangle| = ||v|| \big| ||w|| \big| \sinh \eta(v, w), \tag{2.9}$$

where $\big| ||w|| \big|$ *denotes the absolute value of* $||w||$*.*

Definition 2.1.20. *The unique nonnegative real number* $\eta(v, w)$ *between space-like* v *and positive time-like* w *is called the* ***Lorentz time-like angle*** *between* v *and* w*.*

[19] clearly spells out the relationship between space-like and time-like vectors. For the purposes of this dissertation, it is worth studying how light-like vectors interact with space-like and time-like vectors, too. Namely, it will be useful to know when the Lorentz inner product between a light-like vector and another vector is nonzero.

Lemma 2.1.21. *Let* w *be any light-like vector. Let* v *be a vector independent from* w*. Then* $\langle v, w\rangle \neq 0$ *if one of the following is true:*

1. *Vector* v *is time-like;*
2. *vector* v *is a space-like vector, and* v *and* w *do not span a light-like subspace;*
3. *vector* v *is light-like.*

Proof. This argument is handled case-by case.

Case 1: Vector v *is time-like.* Then by Lemma 2.1.9, $\langle v, w\rangle \neq 0$.

Case 2: Vector v *is space-like, where* v *and* w *do not span a light-like subspace.* Assume to the contrary that $\langle v, w\rangle = 0$. Note that, in general, the span of a space-like vector and a light-like vector is either time-like or light-like. Let V denote the subspace spanned by v and w. Since V is assumed not to be light-like, it must be time-like. This means there is another light-like vector in V of the form $tv + w$, where $t \neq 0$. Thus, $||tv + w||^2 = t^2||v||^2 + 2t\langle v, w\rangle + ||w||^2 = t^2||v||^2 = 0$, which is a contradiction.

Case 3: Vector v is light-like. A lemma in [13] is used when v and w are both positive or both negative. The argument is that if v, w are two linearly independent light-like vectors, they can be written as $v = v_0 \pm |v_0|e_{n+1}$ and $w = w_0 \pm |w_0|e_{n+1}$, where v_0, w_0 are points in $\mathbb{R}^n$. Then $|v| = \sqrt{2}|v_0|$, and $|w| = \sqrt{2}|w_0|$ so that $\langle v, w\rangle = v_0 \cdot w_0 - |v_0||w_0| = v \cdot w - 2|v_0||w_0| < |v||w| - 2|v_0||w_0| = 0$.

On the other hand, assume without loss of generality that v is negative while w is positive. Then $v = v_0 - |v_0|e_{n+1}$, and $w = w_0 + |w_0|e_{n+1}$, so $\langle v, w\rangle = v_0 \cdot w_0 + |v_0||w_0| = v \cdot w > 0$. □

It is possible that $\langle v, w\rangle = 0$ for space-like v and light-like w. The above observation implies that if v is a positive or negative space-like vector and w is a light-like vector where $\langle v, w\rangle = 0$, then the vectors must span a light-like subspace. Here is an example of this in $\mathbb{R}^3$: Let $v = (2, 1, 2)$, and let $w = (1, 0, 1)$. The subspace V spanned by v and w must be light-like because for all vectors in V that aren't scalar multiplies of v or w have Lorentz norm $||tv + w||^2 = t^2 > 0$, where $t \neq 0$.

2.2 Lorentz Transformations

Definition 2.2.1. *A **Lorentz transformation** is a linear map of $\mathbb{R}^{n+1}$ that preserves the Lorentz inner product. That is, for vectors $v = [v_1, \ldots, v_{n+1}], w = [w_1, \ldots, w_{n+1}]$ in $\mathbb{R}^{n+1}$, and $(n+1) \times (n+1)$ matrix A representing a linear map, A is a Lorentz transformation if $\langle vA, wA\rangle = \langle v, w\rangle$, where vA is matrix multiplication. A Lorentz transformation is **positive** (resp. **negative**) if it takes positive time-like vectors to positive (resp. negative) time-like vectors.*

Consider the set of Lorentz transformations, denoted $O(n, 1) = \{A \in M(n+1) : \langle vA, wA\rangle = \langle v, w\rangle\}$. This set is a group under matrix multiplication, known as the **Lorentz group**.

Let $[B]$ be the matrix in (6.2). For v, w in $\mathbb{R}^{n+1}$, $vBw^t = \langle v, w\rangle$. Then $\langle vA, wA\rangle = \langle v, w\rangle$ implies that $vABA^tw^t = vBw^t$ for all v, w. So, equivalently, the Lorentz group is written $O(n, 1) = \{A \in M(n+1) : ABA^t = B\}$. Note that $\det(ABA^t) = \det(A)\det(B)\det(A^t) = \det(A)\det(B)\det(A) = \det(B)$ so $[\det(A)]^2 = 1$, meaning $\det(A) \in \{\pm 1\}$. This means that A is invertible, and specifically, that $O(n, 1)$ is the set of $(n+1) \times (n+1)$ ***orthogonal matrices***, such that $A^{-1} = A^t$, for any

A in $O(n,1)$. The subset of Lorentz transformations with $\det(A) = 1$ for each A is known as the ***special Lorentz group***, $SO(n,1) = \{A \in O(n,1) : \det(A) = 1\}$, which is the group of orthogonal matrices preserving orientation. The ***positive Lorentz group*** is the subset of Lorentz group $O(n,1)$ restricting to the matrices corresponding to positive Lorentz transformations, denoted $O^+(n,1) = \{A \in O(n,1) : v_{n+1} > 0 \Rightarrow (vA)_{n+1} > 0\}$. The positive Lorentz group is naturally isomorphic to the ***projective Lorentz group***, $O(n,1)/\{\pm I\}$, where I is the identity matrix. Within $SO(n,1)$, the subset of orientation-preserving positive Lorentz transformations, $SO^+(n,1)$ is called the ***positive special Lorentz group***. It should be noted that $O^+(n,1)$ is an index 2 subgroup of $O(n,1)$, and $SO^+(n,1)$ is an index 2 subgroup of $SO(n,1)$.

Theorem 2.2.2 ([19])**.** *For every dimension $m \leq n$, the positive Lorentz group $O^+(n,1)$ acts transitively on:*

1. *The set of m-dimensional time-like subspaces of $\mathbb{R}^{n+1}$;*
2. *The set of m-dimensional space-like subspaces of $\mathbb{R}^{n+1}$;*
3. *The set of m-dimensional light-like subspaces of $\mathbb{R}^{n+1}$.*

Lemma 2.2.3 ([19])**.** *Every A in $O(n,1)$ is either positive or negative.*

2.3 Rigidity of Vectors and Lines in Lorentz n-Space

The word "rigidity" in this context is synonymous with the concept of global uniqueness of a collection. When a configuration satisfying certain conditions is rigid, it means there are no other configurations satisfying the same conditions wherein the movement between the configurations is a non-trivial transformation. When one asks about the uniqueness of a collection of objects, the qualifier is always "unique *up to* what?" In order to allow for the more general configurations of vectors and lines, we will consider uniqueness up to Lorentz transformation. If one wants to consider uniqueness up to positive Lorentz transformation, configurations must be composed of lines and all positive vectors or lines and all negative vectors. At the end of the chapter, a statement is made about what can be said about general configurations up to positive Lorentz transformation.

An integral part of tackling the question of uniqueness of a collection of objects involves asking when one object can be uniquely placed by the information attached to it. In Euclidean geometry, one can uniquely place a point x in $\mathbb{R}^n$ by knowing its distance to an **maximally independent**

subcollection of $n+1$ points. That is, a collection of $n+1$ points that do not lie in a $(n-1)$-dimensional subspace.

When a collection of vectors forms a basis for a space, the maximally linearly independent collection of vectors corresponds to a maximally independent collection of points: the common initial point together with the terminal points. Thus, an analogous statement can be made for vectors in Euclidean space. The following lemma shows that the same is true for vectors in Lorentz space.

Lemma 2.3.1. *Let $\{v_i\}$ be a collection of $n+1$ vectors forming a basis in $\mathbb{R}^{n+1}$. For vectors v and v' not in the basis, if $\langle v, v_i\rangle = \langle v', v_i\rangle$ for each i, then $v = v'$.*

Proof. Let $\{v_i\}$ be a collection of vectors in $\mathbb{R}^{n+1}$ that form a basis for the space. Let v and v' be two vectors distinct from all v_i in the basis. Assume $\langle v, v_i\rangle = \langle v', v_i\rangle$ for each i. Since $\langle\cdot,\cdot\rangle$ is a bilinear form, we get that $\langle v, v_i\rangle - \langle v', v_i\rangle = \langle v - v', v_i\rangle = 0$ for each i.

We set out to show $v - v'$ is not equal to v_j for some j. Assume to the contrary that there is some j so that $v - v' = v_j$. Then v_j is a basis element such that $\langle v_j, v_i\rangle = 0$ for all i. In particular, $\langle v_j, v_j\rangle = 0$, so v_j is light-like. The set $\{v_i : \forall i \neq j\}$ is a basis for an n-dimensional subspace V of $\mathbb{R}^{n+1}$. For any u in V, $u = \Sigma_i b_i v_i$, so $\langle v_j, u\rangle = \langle v_j, \Sigma_i b_i v_i\rangle = \Sigma_i b_i\langle v_j, v_i\rangle = 0$. This makes v_j the Lorentz complement of V. But v_j is light-like, meaning V would also have to be a light-like subspace containing v_j, which is a contradiction.

By Lemma 2.1.3, then $v - v' = 0$, and thus $v = v'$. □

In [13], the rigidity of vectors in Lorentz space uses knowing the Lorentz inner product between *every* pair of vectors in the configuration as a condition. To cut down on the amount of Lorentz inner product information required, the condition is added that the configurations contain a basis for $\mathbb{R}^{n+1}$ within each collection.

Requiring a basis in each collection satisfying the same Lorentz inner product information also serves the purpose of generating the Lorentz transformation between the configurations via the change-of-basis map.

Lemma 2.3.2. *Let $\{v_i\}$ and $\{v'_i\}$ be two collections of vectors, with $i = 1, ..., n+1$, each forming a basis for $\mathbb{R}^{n+1}$. If $\langle v_i, v_j\rangle = \langle v'_i, v'_j\rangle$ for each i, then there is a unique Lorentz transformation Φ such that $\Phi(v_i) = v'_i$ for every i.*

Proof. Let $\{v_i\}$ and $\{v'_i\}$ each be a basis of vectors for $\mathbb{R}^{n+1}$, and assume $\langle v_i, v_j\rangle = \langle v'_i, v'_j\rangle$ for each i. Let Φ be the unique bijective linear map (the change-of-basis map) that satisfies $\Phi(v_i) = v'_i$ for each i. Let $x = \Sigma_i a_i v_i$ and $y = \Sigma_j b_j v_j$ in $\mathbb{R}^{n+1}$. Then $\langle\Phi(x), \Phi(y)\rangle = \langle\Phi(\Sigma_i a_i v_i), \Phi(\Sigma_j b_j v_j)\rangle = \langle\Sigma_i a_i \Phi(v_i), \Sigma_i b_j \Phi(v_j)\rangle = \Sigma_{i,j} a_i b_j \langle\Phi(v_i), \Phi(v_j)\rangle = \Sigma_{i,j} a_i b_j \langle v'_i, v'_j\rangle = \Sigma_{i,j} a_i b_j \langle v_i, v_j\rangle = \langle x, y\rangle$, so Φ is a Lorentz transformation. □

With lemmas 2.3.1 and 2.3.2 in place, we are now ready to assess the rigidity of configurations of vectors and light-like lines over several cases in which varying conditions are applied. The first statement involves using the Lorentz inner product between vectors only. This theorem is used in proving the other cases in which light-like lines are involved and other Lorentz invariants are used. Introducing multiple Lorentz invariants gives the user more than one tool to use with vectors and light-like lines in Lorentz space.

2.3.1 The Rigidity of Vectors Using the Lorentz Inner Product.

In [13], Crane and Short state that a collection of vectors with a maximal amount of Lorentz inner product information known is unique up to Lorentz transformation.

Theorem 2.3.3 (Crane and Short)**.** *Let $\{v_\alpha : \alpha \in \mathcal{A}\}$ and $\{v'_\alpha : \alpha \in \mathcal{A}\}$ be two collections of vectors in $\mathbb{R}^{n+1}$ such that $\langle v_\alpha, v_\beta\rangle = \langle v'_\alpha, v'_\beta\rangle$ for all pairs of α and β in $\mathcal{A}$. Suppose that the subspace spanned by the v_α is either time-like or space-like. Then there is a Lorentz transformation Φ with $\Phi(v_\alpha) = v'_\alpha$ for each α in $\mathcal{A}$.*

In the following theorem, the collections of vectors and subcollections of basis vectors may involve any combination of space-like, time-like, or light-like vectors.

Theorem 2.3.4. *Let $\{v_\alpha : \alpha \in \mathcal{A}\}$ and $\{v'_\alpha : \alpha \in \mathcal{A}\}$ be two collections of distinct vectors in $\mathbb{R}^{n+1}$, indexed by the same set, with at least $n+1$ elements $\{v_i\}$ and $\{v'_i\}$, respectively, that form a basis. Then $\langle v_\alpha, v_i\rangle = \langle v'_\alpha, v'_i\rangle$ for each α, i, if and only if there is a unique Lorentz transformation Φ such that $\Phi(v_\alpha) = v'_\alpha$. for every $\alpha \in \mathcal{A}$.*

Proof. Let $\{v_\alpha : \alpha \in \mathcal{A}\}$ and $\{v'_\alpha : \alpha \in \mathcal{A}\}$ be two collections as described above. The reverse direction of this proof is trivial. Assume $\langle v_\alpha, v_i\rangle = \langle v'_\alpha, v'_i\rangle$ for each i, where $v_i \neq v_\alpha$, $v'_i \neq v'_\alpha$. Since $\{v_i\}$ and $\{v'_i\}$ each form a basis of $\mathbb{R}^{n+1}$, where $\langle v_i, v_j\rangle = \langle v'_i, v'_j\rangle$ for each $i \neq j$, by Lemma 2.3.2, there is a unique Lorentz transformation Φ such that $\Phi(v_i) = v'_i$ for each i. Since Φ is a

Lorentz transformation, observe that $\langle \Phi(v_\alpha), v'_i \rangle = \langle \Phi(v_\alpha), \Phi(v_i) \rangle = \langle v_\alpha, v_i \rangle = \langle v'_\alpha, v'_i \rangle$ for $\alpha \neq i$. So $\langle \Phi(v_\alpha) - v'_\alpha, v'_i \rangle = 0$ for each i, and every $\alpha \neq i$. Thus, by Lemma 2.3.1, we obtain that $\Phi(v_\alpha) - v'_\alpha = 0$, so $\Phi(v_\alpha) = v'_\alpha$ for every α. □

To compare this result with [13], when the collections of vectors is finite of order m, where $m \geq n+1$, Crane and Short require $m(m-1)/2$, Lorentz inner product pairs. With theorem 2.3.4, the amount of required pairs is reduced to $(n+1)(m-(n+1))$.

2.3.2 The Rigidity of Light-Like Lines Using the Absolute Cross Ratio of Lines

In situations where the setting is restricted to the light cone, one may wish to work with light-like lines rather than a specific vector in that line. In this case, one can employ the use of a Lorentz invariant of four light-like lines, defined in [13].

Definition 2.3.5 (Crane and Short). *Let ℓ_1, ℓ_2, ℓ_3, and ℓ_4 be lines in $\mathbb{R}^{n+1}$ through the origin and a point $(x_1, ..., x_n, 1)$, where $x = (x_1, ..., x_n)$ is a point in $\mathbb{S}^{n-1}$ (light-like 1-dimensional subspaces of $\mathbb{R}^{n+1}$). For each ℓ_i, choose a vector v_i in ℓ_i. The **absolute cross ratio of lines**, denoted $|\ell_1, \ell_2, \ell_3, \ell_4|$, is defined to be*

$$|\ell_1, \ell_2, \ell_3, \ell_4| = \frac{\langle v_1, v_3 \rangle \langle v_2, v_4 \rangle}{\langle v_1, v_2 \rangle \langle v_3, v_4 \rangle}. \tag{2.10}$$

The absolute cross ratio of lines is clearly a Lorentz invariant of the ordered 4-tuple. It is also independent of the choice of light-like v_i in ℓ_i for each $1 \leq i \leq 4$. To see this, let $\lambda_i v_i$ be any other nonzero vector in ℓ_i, where λ_i is some nonzero real number. Then

$$|\ell_1, \ell_2, \ell_3, \ell_4| = \frac{\langle \lambda_1 v_1, \lambda_3 v_3 \rangle \langle \lambda_2 v_2, \lambda_4 v_4 \rangle}{\langle \lambda_1 v_1, \lambda_2 v_2 \rangle \langle \lambda_3 v_3, \lambda_4 v_4 \rangle} = \frac{\lambda_1 \lambda_3 \lambda_2 \lambda_4 \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle}{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \langle v_1, v_2 \rangle \langle v_3, v_4 \rangle} = \frac{\langle v_1, v_3 \rangle \langle v_2, v_4 \rangle}{\langle v_1, v_2 \rangle \langle v_3, v_4 \rangle}. \tag{2.11}$$

Note also that the absolute cross ratio of lines is indeed a positive value, because $\langle v_i, v_j \rangle < 0$ for each i, j pair. Consider the following theorem from [13].

Theorem 2.3.6 (Crane and Short). *Given two collections of light-like lines $\{\ell_\alpha : \alpha \in \mathcal{A}\}$ and $\{\ell'_\alpha : \alpha \in \mathcal{A}\}$, there is a positive Lorentz transformation Φ with $\Phi(\ell_\alpha) = \ell'_\alpha$ for each α in $\mathcal{A}$ if and only if $|\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_\sigma| = |\ell'_\alpha, \ell'_\beta, \ell'_\gamma, \ell'_\sigma|$ for all ordered 4-tuples $(\alpha, \beta, \gamma, \sigma)$ of distinct indices in $\mathcal{A}$.*

The following theorem is a direct result of modifying Theorem 2.3.6. Crane and Short utilize the absolute cross ratio between every 4-tuple of lines, but, using theorem 2.3.4, one can trim down the number of 4-tuples assessed.

Since we are assessing the rigidity of lines instead of vectors in this section, we are back to considering maximally independent collections of lines rather than a basis of vectors.

Definition 2.3.7. *A collection of lines $\{\ell_i\}$ of size $n+1$ in $\mathbb{R}^{n+1}$ is* ***maximally independent*** *if the collection $\{v_i\}$ of vectors, where $v_i \neq 0$ is in ℓ_i for each i, is a basis for $\mathbb{R}^{n+1}$.*

Theorem 2.3.8. *Let $\{\ell_\alpha : \alpha \in \mathcal{A}\}$ and $\{\ell'_\alpha : \alpha \in \mathcal{A}\}$ be two collections of distinct light-like lines in $\mathbb{R}^{n+1}$, each with subcollections of $n+1$ lines $\{\ell_i\}$ and $\{\ell'_i\}$, respectively, that are maximally independent in $\mathbb{R}^{n+1}$. Then,*

$$|\ell_\alpha, \ell_i, \ell_j, \ell_k| = |\ell'_\alpha, \ell'_i, \ell'_j, \ell'_k|,$$

for every distinct triplet (i,j,k) in the independent subcollection index, and all α, if and only if there is a unique positive Lorentz transformation Φ such that $\Phi(\ell_\alpha) = \ell'_\alpha$, for all $\alpha \in \mathcal{A}$.

Proof. The converse direction of the statement is trivial. Assume $|\ell_\alpha, \ell_i, \ell_j, \ell_k| = |\ell'_\alpha, \ell'_i, \ell'_j, \ell'_k|$, for every distinct triplet (i,j,k) in the independent subcollection index, and all α. Each ℓ_α will be represented with a chosen light-like vector v_α in ℓ_α. Choose v_1, v_2, v_3 and v'_1, v'_2, v'_3 so that $\langle v_1, v_2\rangle = \langle v'_1, v'_2\rangle$, $\langle v_1, v_3\rangle = \langle v'_1, v'_3\rangle$, and $\langle v_2, v_3\rangle = \langle v'_2, v'_3\rangle$. Choose v_α such that $\langle v_\alpha, v_2\rangle = \dfrac{-\langle v_2, v_3\rangle}{\langle v_1, v_3\rangle}$, and similarly, $\langle v'_\alpha, v'_2\rangle = \dfrac{-\langle v'_2, v'_3\rangle}{\langle v'_1, v'_3\rangle}$, so that $\langle v_\alpha, v_2\rangle = \langle v'_\alpha, v'_2\rangle$. Then, since $|v_\alpha, v_1, v_2, v_3| = |v'_\alpha, v'_1, v'_2, v'_3|$, we get that $\langle v_\alpha, v_1\rangle = \langle v'_\alpha, v'_1\rangle$. Now, $|v_\alpha, v_i, v_j, v_k| = |v'_\alpha, v'_i, v'_j, v'_k|$, for all distinct (i,j,k) in the independent subcollection index, so in particular, $|v_\alpha, v_i, v_1, v_2| = |v'_\alpha, v'_i, v'_1, v'_2|$, meaning

$$\frac{\langle v_\alpha, v_1\rangle\langle v_i, v_2\rangle}{\langle v_\alpha, v_i\rangle\langle v_1, v_2\rangle} = \frac{\langle v'_\alpha, v'_1\rangle\langle v'_i, v'_2\rangle}{\langle v'_\alpha, v'_i\rangle\langle v'_1, v'_2\rangle}, \tag{2.12}$$

for all $i \neq 1, 2, \alpha$. By design, $\langle v_1, v_2\rangle = \langle v'_1, v'_2\rangle$, and $\langle v_i, v_2\rangle = \langle v'_i, v'_2\rangle$, so $\langle v_\alpha, v_i\rangle = \langle v'_\alpha, v'_i\rangle$, for all α, and all i. Applying theorem 2.3.4, there is a Lorentz transformation Φ such that $\Phi(v_\alpha) = v'_\alpha$ and consequently $\Phi(\ell_\alpha) = \ell'_\alpha$ for all α. Either Φ is positive, or $-\Phi$ is positive. If Φ is positive, then we're done. If $-\Phi$ is positive, $-\Phi(v_\alpha) = -v'_\alpha$ for all α, and so it is still true that $-\Phi(\ell_\alpha) = \ell'_\alpha$ for all α. □

The absolute cross ratio of lines is used here because it translates to a statement involving the absolute cross ratio of points (seen in the proceeding chapter). This is the information that is commonly used as a conformal invariant of points, so we maintain using this invariant within the context of the above proof.

2.3.3 Rigidity of Space-Like Vectors, Time-Like Vectors and Light-Like Lines Using the Lorentz Ratio

This section shows that the absolute cross ratio of lines can be modified to produce another kind of Lorentz invariant that can be used more generally with light-like lines and space-like or time-like vectors. There is a secondary motive for introducing another Lorentz invariant outside the context of Lorentz Geometry. This Lorentz invariant is introduced because the Lorentz inner product of a light-like vector and space-like vector or time-like vector does not correspond to geometric information between spheres and points in $\mathbb{S}^{n-1}$. In this section, we view light-like lines as the limit of a sequence of space-like or time-like unit vectors as a means to remedy this issue.

Definition 2.3.9. *Let $\{v_t\}$ be a sequence of all positive (negative) space-like, or all positive (negative) time-like, unit vectors. Let ℓ be a light-like line, and let w_ℓ be any positive (negative) light-like vector in ℓ. Then $\{v_t\}$* ***converges to** ℓ **as*** *$t \to \infty$ if for every $\epsilon > 0$, there is an $N > 0$ such that for all $t \geq N$, $|\measuredangle(v_t, w_\ell)| < \epsilon$, where $\measuredangle(v_t, w_\ell)$ denotes the Euclidean angle between w_ℓ and each v_t.*

Several observations can be made from this definition.

Observation 1. Let $\{v_t\}$ be a sequence following the conditions in definition 2.3.9, converging towards a light-like line ℓ. Let w_ℓ be any positive vector in ℓ. Say w_ℓ has component form $w_\ell = (w_{\ell 1}, \ldots, w_{\ell n}, w_{\ell(n+1)})$. Then for each t, v_t can always be written as

$$v_t = \left(\sqrt{\lambda_1^2(t) w_{\ell 1}^2 \pm 1}, \lambda_2(t) w_{\ell 2}, \ldots, \lambda_n(t) w_{\ell n}, \lambda_{n+1}(t) w_{\ell(n+1)}\right), \tag{2.13}$$

where $\lambda_1^2(t) w_{\ell 1}^2 + \ldots + \lambda_n^2(t) w_{\ell n}^2 - \lambda_{n+1}^2 w_{\ell(n+1)} = 0$ for each t, and ± 1 in the first coordinate depends on whether v_t is a sequence of space-like or time-like vectors. Moreover, $|\lambda_i(t)| \to \infty$ and $\dfrac{\lambda_i(t)}{\lambda_j(t)} \to 1$ as $t \to \infty$ for each i, j. This is because $\dfrac{v_t \cdot w_\ell}{|v_t||w_\ell| \cos\theta} \to 1$ as $\theta \to 0$, where $\theta = \measuredangle(v_t, w_\ell)$.

Observation 2. Let $w_\ell(t)$ be a sequence of vectors in ℓ such that

$$w_\ell(t) = (\lambda(t) w_{\ell 1}, \ldots, \lambda(t) w_{\ell n}, \lambda(t) w_{\ell(n+1)}), \tag{2.14}$$

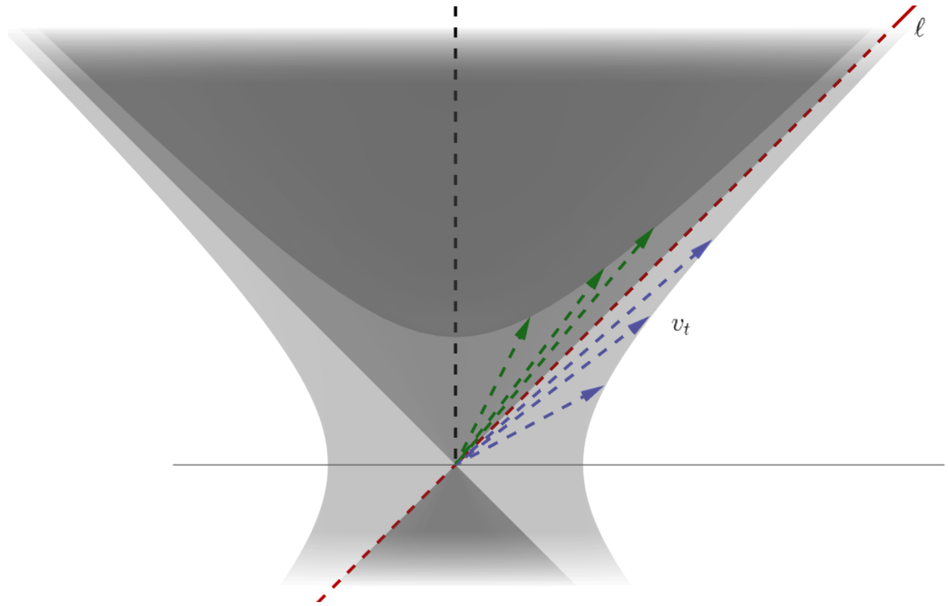

Figure 2.2: The vectors in $\{v_t\}$ are required to be unit vectors so that each vector's terminal point lies in the hyperboloid outside C^n if the v_t are space-like (shown in blue), and inside C^n if the v_t are time-like (shown in green). With this being the case, for light-like line ℓ (shown in red), as $\measuredangle(v_t, \ell)$ converges to 0, Observation 3 holds.

where $\lambda(t) = \lambda_{n+1}(t)$ for each t. As a result, $\lambda_i(t) \to \lambda(t)$ for each i as $t \to \infty$, so the vector $v_t - w_\ell(t) \to 0$ as $t \to \infty$.

Observation 3. Let u be a positive (negative) vector and let ℓ be a light-like line, where the 2-dimensional subspace spanned by u and ℓ is not light-like. Let v_t be a sequence of all space-like or all time-like unit vectors converging to the light-like line ℓ. Then as $v_t \to \ell$, $|\langle v_t, u\rangle| \to \infty$.

Let v_ℓ be some light-like vector in ℓ. Let $w_\ell(t)$ be the sequence of vectors in ℓ with the same scalar in the $(n+1)$-coordinate as v_t for each t. Let $\lambda(t)$ be the sequence of real nonzero scalars such that $w_\ell(t) = \lambda(t)v_\ell$ for each t. As $v_t \to \ell$, since $v_t - w_\ell(t) \to 0$, we get that $|\langle v_t, u\rangle| \to |\langle w_\ell(t), u\rangle| = |\lambda(t)\langle v_\ell, u\rangle|$, where $|\lambda(t)| \to \infty$ as $t \to \infty$. Since by Lemma 2.1.21, $\langle v_\ell, u\rangle \neq 0$, it is concluded that $|\lambda(t)\langle v_\ell, u\rangle| \to \infty$.

The following is the Lorentz invariant introduced for the purposes of comparing light-like lines to vectors.

Definition 2.3.10. *Let $\{v_t\}$ be a sequence of all positive (negative) space-like, or all positive (negative) time-like unit vectors converging to a light-like line ℓ. Then for any two vectors u_1, u_2 such* *that the subspace spanned by ℓ and u_i is not light-like for each i, the* ***Lorentz ratio of ℓ, u_1, and u_2****, denoted (ℓ, u_1, u_2), is*

$$(\ell, w_1, w_2) = \lim_{t\to\infty} \frac{\langle v_t, u_1\rangle}{\langle v_t, u_2\rangle}. \tag{2.15}$$

Observation 4. The Lorentz ratio (ℓ, u_1, u_2) can be positive or negative.

The Lorentz ratio will be useful as defined within the geometry of spheres and points in $\mathbb{S}^{n-1}$, but as it stands, it is an awkward measurement to require when only considering vector space information. The next lemma provides a convenient observation about the Lorentz ratio.

Lemma 2.3.11. *Let $\{v_t\}$ be a sequence of all positive (negative) space-like, or all positive (negative) time-like, unit vectors such that v_t converges to light-like line ℓ as $t \to \infty$. Let v_ℓ be any vector in ℓ. Then*

$$\lim_{t\to\infty} \frac{\langle v_t, u_1\rangle}{\langle v_t, u_2\rangle} = \frac{\langle v_\ell, u_1\rangle}{\langle v_\ell, u_2\rangle} \tag{2.16}$$

for any two vectors u_1, u_2, where the subspace spanned by ℓ and u_i is not light-like for each i.

Proof. Let ℓ be a light-like line, let v_ℓ be any light-like vector in ℓ, and without loss of generality. let $\{v_t\}$ be a sequence of positive (negative) space-like unit vectors such that $v_t \to \ell$ as $t \to \infty$. The following argument is still valid if $\{v_t\}$ is instead a sequence of positive (negative) time-like unit vectors limiting toward line ℓ. Let u_1, u_2 be two vectors such that the subspace spanned by ℓ and u_i is not light-like for each i.

Let $\epsilon > 0$. First note that if there is an $N > 0$ such that for all $t \geq N$,

$$|\langle v_t, u_1\rangle\langle v_\ell, u_2\rangle - \langle v_\ell, u_1\rangle\langle v_t, u_2\rangle| < \epsilon$$

then the statement is proven, because

$$\left|\frac{\langle v_t, u_1\rangle}{\langle v_t, u_2\rangle} - \frac{\langle v_\ell, u_1\rangle}{\langle v_\ell, u_2\rangle}\right| = \left|\frac{\langle v_t, u_1\rangle\langle v_\ell, u_2\rangle - \langle v_\ell, u_1\rangle\langle v_t, u_2\rangle}{\langle v_t, u_2\rangle\langle v_\ell, u_2\rangle}\right|,$$

and as $t \to \infty$, $|\langle v_t, u_2\rangle\langle v_\ell, u_2\rangle| \to \infty$.

Let $w_\ell(t)$ be the sequence of light-like vectors in ℓ, parametrized by t, such that $w_{\ell(n+1)}(t) = v_{t(n+1)}$ for each t. Since $v_t \to \ell$ as $t \to \infty$, for every $\epsilon > 0$, there exists $N > 0$ such that for all $t \geq N$,

$|v_t - w_\ell(t)| < \epsilon$. Consequently, for every $\epsilon > 0$, there is an $N_i > 0$ for each $i = 1, 2$ such that for all $t \geq N_i$, $|\langle v_t, u_i\rangle - \langle w_\ell(t), u_i\rangle| = |\langle v_t - w_\ell(t), u_i\rangle| < \epsilon$. In particular, let $\epsilon_0 = \dfrac{\epsilon}{|\langle v_\ell, u_2\rangle| + |\langle v_\ell, u_1\rangle|}$. Let $N_0 = \max\{N_1, N_2\}$ so that for all $t \geq N_0$, $|\langle v_t, u_i\rangle - \langle w_\ell(t), u_i\rangle| < \dfrac{\epsilon}{|\langle v_\ell, u_2\rangle| + |\langle v_\ell, u_1\rangle|}$ for $i = 1, 2$.

For each t, correct v_ℓ by $\lambda(t)$ so that $\lambda(t)v_\ell = w_\ell(t)$.

Then for all $t \geq N_0$,

$$\begin{aligned}
|\langle v_t, u_1\rangle\langle v_\ell, u_2\rangle - \langle v_\ell, u_1\rangle\langle v_t, u_2\rangle| &= \frac{1}{|\lambda(t)|}\,|\langle v_t, u_1\rangle\langle w_\ell(t), u_2\rangle - \langle w_\ell(t), u_1\rangle\langle v_t, u_2\rangle| \\
&= \frac{1}{|\lambda(t)|}|\langle v_t, u_1\rangle\langle w_\ell(t), u_2\rangle - \langle w_\ell(t), u_1\rangle\langle w_\ell(t), u_2\rangle \\
&\quad +\langle w_\ell(t), u_1\rangle\langle w_\ell(t), u_2\rangle - \langle w_\ell(t), u_1\rangle\langle v_t, u_2\rangle| \\
&\leq \frac{1}{|\lambda(t)|}(|\langle v_t, u_1\rangle\langle w_\ell(t), u_2\rangle - \langle w_\ell(t), u_1\rangle\langle w_\ell(t), u_2\rangle| \\
&\quad + |\langle w_\ell(t), u_1\rangle\langle w_\ell(t)u_2\rangle - \langle w_\ell(t), u_1\rangle\langle v_t, u_2\rangle|) \\
&= \frac{1}{|\lambda(t)|}(|\langle w_\ell(t), u_2\rangle|\,|\langle v_t, u_1\rangle - \langle w_\ell(t), u_1\rangle| \\
&\quad + |\langle w_\ell(t), u_1\rangle|\,|\langle v_t, u_2\rangle - \langle w_\ell(t), u_2\rangle|) \\
&< \frac{1}{|\lambda(t)|}\Bigg(|\langle w_\ell(t), u_2\rangle|\,\frac{\epsilon}{|\langle v_\ell, u_2\rangle| + |\langle v_\ell, u_1\rangle|} \\
&\quad + |\langle w_\ell(t), u_1\rangle|\,\frac{\epsilon}{|\langle v_\ell, u_2\rangle| + |\langle v_\ell, u_1\rangle|}\Bigg) \\
&= \left(\frac{\left|\langle \frac{w_\ell(t)}{\lambda(t)}, u_2\rangle\right|}{|\langle v_\ell, u_2\rangle| + |\langle v_\ell, u_1\rangle|} + \frac{\left|\langle \frac{w_\ell(t)}{\lambda(t)}, u_1\rangle\right|}{|\langle v_\ell, u_2\rangle| + |\langle v_\ell, u_1\rangle|}\right)\epsilon \\
&= \left(\frac{|\langle v_\ell, u_2\rangle|}{|\langle v_\ell, u_2\rangle| + |\langle v_\ell, u_1\rangle|} + \frac{|\langle v_\ell, u_1\rangle|}{|\langle v_\ell, u_2\rangle| + |\langle v_\ell, u_1\rangle|}\right)\epsilon \\
&= \epsilon.
\end{aligned}$$

□

Using equality (2.16), it is clear that the Lorentz ratio is a Lorentz invariant, since for any Lorentz transformation T, light-like line ℓ, and space-like or time-like vectors w_1, w_2, observe that $(T(\ell), T(w_1), T(w_2)) = \dfrac{\langle T(v_\ell), T(w_1)\rangle}{\langle T(v_\ell), T(w_2)\rangle} = \dfrac{\langle v_\ell, w_1\rangle}{\langle v_\ell, w_2\rangle} = (\ell, w_1, w_2)$.

Employing the Lorentz ratio, we now have an alternate means of determining the rigidity of intermingled collections of vectors and light-like lines in Lorentz Space. The following theorem was crafted with the intention of interpreting it geometrically as the most general version of a rigidity statement for intermingled collections of spheres and points in $\mathbb{S}^{n-1}$. Some Corollaries are stated at the end that are also Corollaries of the other main theorems.

Theorem 2.3.12. *Let $\{v_\alpha, v_\beta, \ell_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ and $\{v'_\alpha, v'_\beta, \ell'_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ be two collections of distinct space-like vectors, time-like vectors, and light-like lines, respectively, in $\mathbb{R}^{n+1}$, where each of $\ell_\gamma, \ell'_\gamma$, respectively do not span a light-like subspace with any of v_α, v'_α or v_β, v'_β, and with at least $n+1$ space-like (or time-like) vectors $\{v_i\}$ and $\{v'_i\}$, respectively, that form a basis for $\mathbb{R}^{n+1}$. Then,*

$$\langle v_\alpha, v_i \rangle = \langle v'_\alpha, v'_i \rangle, \langle v_\beta, v_i \rangle = \langle v'_\beta, v'_i \rangle,$$

for each i, for all space-like vectors v_α, v'_α, for all time-like vectors v_β, v'_β, and $(\ell_\gamma, v_i, v_j) = (\ell'_\gamma, v'_i, v'_j)$, for each distinct pair i, j in the independent subcollection index, and all light-like $\ell_\gamma, \ell'_\gamma$ if and only if there is a unique Lorentz transformation Φ such that $\Phi(v_\alpha) = v'_\alpha$, $\Phi(v_\beta) = v'_\beta$, and $\Phi(\ell_\gamma) = \ell'_\gamma$ for all $\alpha, \beta, \gamma \in \mathcal{A}$.

Proof. Let $\{v_\alpha, v_\beta, \ell_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ and $\{v'_\alpha, v'_\beta, \ell'_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ be two collections of space-like and time-like vectors, and light-like lines, with respective independent subcollections of space-like (or time-like) vectors $\{v_i\}$, $\{v'_i\}$, each forming a basis of $\mathbb{R}^{n+1}$. If there is a Lorentz transformation Φ such that $\Phi(v_\alpha) = v'_\alpha, \Phi(v_\beta) = v'_\beta$, and $\Phi(\ell_\gamma) = \ell'_\gamma$, then trivially, $\langle v_\alpha, v_i \rangle = \langle v'_\alpha, v'_i \rangle$, $\langle v_\beta, v_i \rangle = \langle v'_\beta, v'_i \rangle$, and $(\ell_\gamma, v_i, v_j) = (\ell'_\gamma, v'_i, v'_j)$, for each distinct pair i, j, and all $\alpha, \beta, \gamma \in \mathcal{A}$. Assume, conversely, that $\langle v_\alpha, v_i \rangle = \langle v'_\alpha, v'_i \rangle$, $\langle v_\beta, v_i \rangle = \langle v'_\beta, v'_i \rangle$, and $(\ell_\gamma, v_i, v_j) = (\ell'_\gamma, v'_i, v'_j)$, for each distinct pair i, j, and all $\alpha, \beta, \gamma \in \mathcal{A}$. The only work to be done is to show the assumption that $(\ell_\gamma, v_i, v_j) = (\ell'_\gamma, v'_i, v'_j)$ implies $\langle v_\gamma, v_i \rangle = \langle v'_\gamma, v'_i \rangle$ for each i, and chosen v_γ, v'_γ in all $\ell_\gamma, \ell'_\gamma$ respectively. Choose v_γ in each ℓ_γ and v'_γ in each ℓ'_γ such that $\langle v_\gamma, v_1 \rangle = \langle v'_\gamma, v'_1 \rangle$. This can always be done. Using our assumptions for $j = 1$, and $i \neq 1$, $(\ell_\gamma, v_i, v_1) = (\ell'_\gamma, v'_i, v'_1)$ means $\dfrac{\langle v_\gamma, v_i \rangle}{\langle v'_\gamma, v_1 \rangle} = \dfrac{\langle v'_\gamma, v'_i \rangle}{\langle v'_\gamma, v'_1 \rangle}$, so using theorem 2.3.4, there is a Lorentz transformation Φ such that $\Phi(v_\alpha) = v'_\alpha, \Phi(v_\beta) = v'_\beta$, and $\Phi(v_\gamma) = v'_\gamma$, which by extension means $\Phi(\ell_\gamma) = \ell'_\gamma$ for all $\alpha, \beta, \gamma \in \mathcal{A}$. □

Corollary 2.3.13. *In the set up of the previous statement, Φ is either a positive Lorentz transformation, or $-\Phi$ is a unique positive Lorentz transformation such that $-\Phi(v_\alpha) = -v'_\alpha, -\Phi(v_\beta) = -v'_\beta$, and $-\Phi(\ell_\gamma) = \ell'_\gamma$.*

Corollary 2.3.14. *Let $\{v_\alpha, v_\beta, \ell_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ and $\{v'_\alpha, v'_\beta, \ell'_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ be two collections of space-like vectors, time-like vectors and light-like lines, respectively, in $\mathbb{R}^{n+1}$, where each of ℓ_γ and*

ℓ'_γ respectively do not span a light-like subspace with any of v_α, v'_α or v_β, v'_β. Suppose each collection has a subcollection of the same order, $\{v_i\}$ and $\{v'_i\}$ that is maximally linearly independent in the collection, where $\langle v_\alpha, v_i\rangle = \langle v'_\alpha v'_i\rangle$ for each distinct i, α, and $(\ell_\gamma, v_i, v_j) = (\ell'_\gamma, v'_i, v'_j)$ for each distinct triple γ, i, j. Then if $\{v_i\}$ and $\{v'_i\}$ span a time-like or space-like subspace, there is a Lorentz transformation ϕ such that $\phi(v_\alpha) = v'_\alpha$, $\phi(v_\beta) = v'_\beta$, and $\phi(\ell_\gamma) = \ell'_\gamma$ for every α, β, γ in $\mathcal{A}$.

This last corollary is an observation based upon the fact that space-like and time-like subspaces are non-degenerate. If one collection is time-like (resp. space-like), then necessarily, the other collection is time-like (resp. space-like). By Theorem 2.2.2, the positive Lorentz transformations act transitively on m-dimensional time-like and space-like subspaces. Note that the main theorem provides a *unique* Lorentz transformation, while for the corollary, it is possible to have more than one Lorentz transformation satisfying the statement.

CHAPTER 3

POINTS, IDEAL POINTS, AND HYPERPLANES OF HYPERBOLIC N-SPACE

In chapter 2, we explored the geometry of Lorentz Space, independent of external motivations. One reason for this is to observe that Lorentz Space, on its own, is a rich setting where geometric results are handled easily through Linear Algebra. The secondary motivation is to study geometric statements in Lorentz Space. While hyperbolic geometry can be set up and considered as a standalone geometry, there is much insight to be gained by considering hyperbolic space within the context of Lorentz Space.

In this chapter a dictionary is set up between the language of $(n+1)$-dimensional Lorentz Space and the language of hyperbolic n-space. A good reference for this is [19]. What isn't included in [19] is a characterization of how collections of objects in hyperbolic n-space behave when they correspond to a basis of vectors in $\mathbb{R}^{n+1}$. We fill this information in after the foundation is laid. Simple but vital observations are pieced together to craft a rigidity result for points, ideal points and hyperplanes of hyperbolic n-space from the main result in the previous chapter.

3.1 Hyperboloid Model of Hyperbolic n-Space in $\mathbb{R}^{n+1}$

Consider the set of points

$\mathcal{H}^n = \{x \in \mathbb{R}^{n+1} : ||x||^2 = -1, x_{n+1} > 0\}$.

This set describes the *positive* sheet of an n-dimensional hyperboloid $\mathcal{F}^n = \{x \in \mathbb{R}^{n+1} : ||x||^2 = -1\}$ in $\mathbb{R}^{n+1}$, centered at the origin. Let x, y be two points in $\mathcal{H}^n$. Note that x and y can be thought of as positive time-like unit vectors in $(n+1)$-dimensional Lorentz space, and every positive time-like unit vector in $\mathbb{R}^{n+1}$ represents a point in $\mathcal{H}^n$. Let $\eta(x, y)$ be the Lorentz time-like angle between x and y. Then the ***hyperbolic distance between*** x ***and*** y can be defined as

$$d_{\mathcal{H}}(x, y) = \eta(x, y),$$

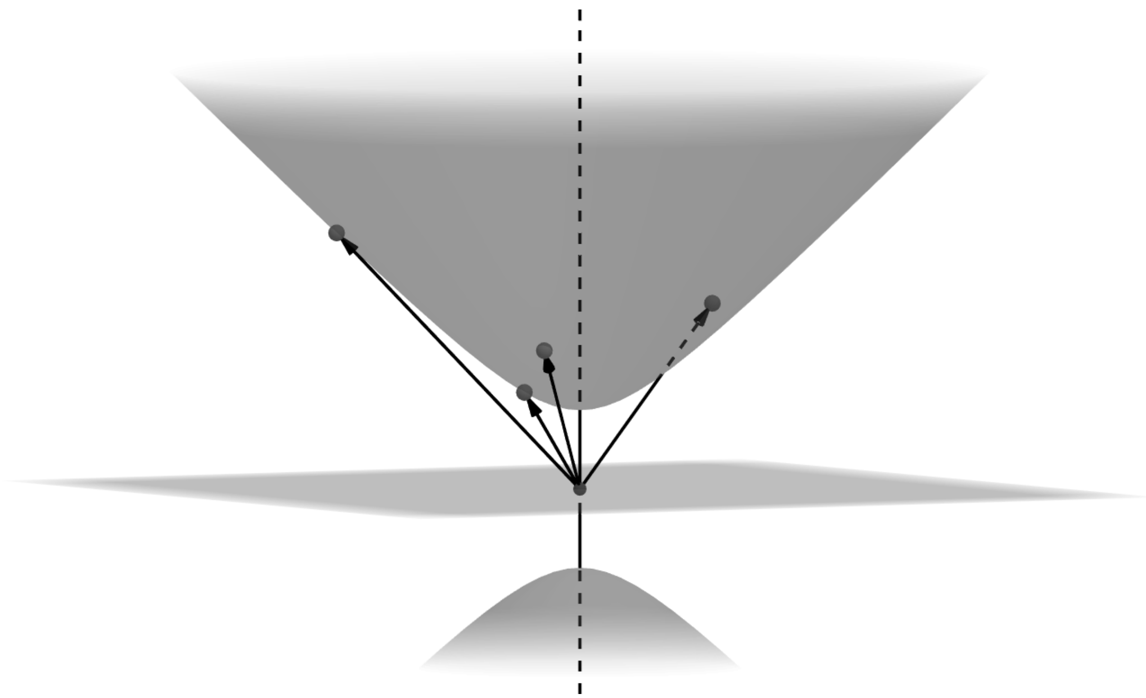

Figure 3.1: Points in the hyperboloid model of hyperbolic space, $\mathcal{H}^2$, and corresponding time-like vectors in 3-dimensional Lorentz space.

so that $\langle x, y\rangle = -\cosh d_{\mathcal{H}}(x, y)$. The set $\mathcal{H}^n$, together with $d_{\mathcal{H}}(\cdot,\cdot)$ is ***the hyperboloid model of hyperbolic n-space***. For a proof that $d_{\mathcal{H}}(\cdot,\cdot)$ is the *hyperbolic metric* on $\mathcal{H}^n$, see [19], theorem 3.2.2.

This connection between positive time-like unit vectors $\mathbb{R}^{n+1}$ and points in hyperbolic n-space is our first established correspondence between a geometric object in $\mathcal{H}^n$ and a vector space object in $\mathbb{R}^{n+1}$. We continue with this endeavor throughout the section. Our next focus is on the transformations of each setting.

3.1.1 Isometries of Hyperbolic n-space

From Chapter 2, we know that the positive Lorentz transformations are an index 2 subgroup of the Lorentz transformations, and that all Lorentz transformations are either positive or negative. Considering that $\mathcal{H}^n$ is the positive sheet of $\mathcal{F}^n$, and using theorem 2.2.2, we can now make the following statement.

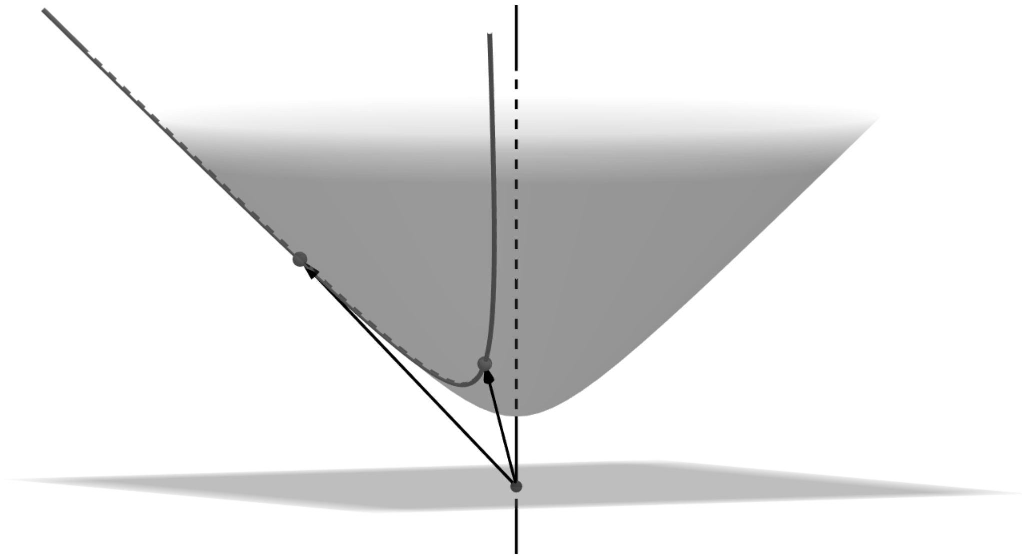

Figure 3.2: Two points in determine a hyperbolic line; this corresponds to two linearly independent positive time-like unit vectors determining a time-like subspace. This time-like subspace intersects the hyperboloid model as a hyperbolic line.

Theorem 3.1.1 ([19])**.** *Every positive Lorentz transformation of* $\mathbb{R}^{n+1}$ *restricts to an isometry of* $\mathcal{H}^n$ *and every isometry of* $\mathcal{H}^n$ *extends to a unique positive Lorentz transformation of* $\mathbb{R}^{n+1}$*.*

Corollary 3.1.2. *The group of hyperbolic isometries* $I(\mathcal{H}^n)$ *is isomorphic to the positive Lorentz group* $O^+(n,1)$*.*

Corollary 3.1.3. *The positive special Lorentz group* $SO^+(n,1)$ *is isomorphic to the group of orientation-preserving isometries of* $\mathcal{H}^n$*, denoted* $I^+(\mathcal{H}^n)$*.*

With this fact in place, we can view hyperbolic geometry under the lens of Lorentz geometry, where the isometric invariants of $\mathcal{H}^n$ can be expressed in formulas involving Lorentz inner products.

3.1.2 Hyperbolic Lines

Definition 3.1.4. *A* ***hyperbolic line*** *of* $\mathcal{H}^n$ *is the intersection of* $\mathcal{H}^n$ *with a 2-dimensional time-like vector subspace of* $\mathbb{R}^{n+1}$*.*

For two points x *and* y *in* $\mathcal{H}^n$*, the span of* x *and* y *is a 2-dimensional time-like subspace* $V(x,y)$ *of* $\mathbb{R}^{n+1}$*, and*

$$L(x,y) = \mathcal{H}^n \cap V(x,y) \tag{3.1}$$

is the unique hyperbolic line of $\mathcal{H}^n$ containing both x and y. The intersection $L(x, y)$ is a branch of a hyperbola.

Definition 3.1.5. *Three points x, y, z in $\mathcal{H}^n$ are **hyperbolically collinear** if and only if there is a hyperbolic line L of $\mathcal{H}^n$ containing x, y, z.*

Lemma 3.1.6 ([19])**.** *If x, y, z are points of $\mathcal{H}^n$ and*

$$\eta(x, y) + \eta(y, z) = \eta(x, z), \tag{3.2}$$

then x, y, z are hyperbolically collinear.

Theorem 3.1.7 ([19])**.** *A function $\lambda : \mathbb{R} \to \mathcal{H}^n$ is a geodesic line if and only if there are Lorentz orthonormal vectors x, y in $\mathbb{R}^{n+1}$ such that*

$$\lambda(t) = (\cosh t)x + (\sinh t)y \tag{3.3}$$

Theorem 3.1.8 ([19])**.** *The geodesics of $\mathcal{H}^n$ are its hyperbolic lines.*

Definition 3.1.9. *A **tangent vector to** $\mathcal{H}^n$ **at a point** x **of** $\mathcal{H}^n$ is defined to be the derivative at 0 of a differentiable curve $\gamma : [-b, b] \to \mathcal{H}^n$ such that $\gamma(0) = x$. The set of all tangent vectors to $\mathcal{H}^n$ at x is called the **tangent space of** $\mathcal{H}^n$ **at** x, and is denoted $T_x = T_x(\mathcal{H}^n)$.*

Lemma 3.1.10 ([19])**.** *Let $T_x = T_x(\mathcal{H}^n)$ be the set of all tangent vectors to $\mathcal{H}^n$ at x. Then*

$$T_x = \{y \in \mathbb{R}^{n+1} : \langle x, y \rangle = 0\}. \tag{3.4}$$

From the above lemma, the tangent space T_x of a given x in $\mathcal{H}^n$ is n-dimensional and space-like in $\mathbb{R}^{n+1}$. That is, $T_x = \langle x \rangle^L$, where $\langle x \rangle^L$ is the Lorentz complement of the subspace spanned by x.

Definition 3.1.11. *Let $\lambda : \mathbb{R} \to \mathcal{H}^n$ and and $\mu : \mathbb{R} \to \mathcal{H}^n$ be geodesic lines such that $\lambda(0) = \mu(0)$. Then $\lambda'(0)$ and $\mu'(0)$ span a space-like vector subspace of $\mathbb{R}^{n+1}$. The **hyperbolic angle** between λ and μ is the Lorentz space-like angle between $\lambda'(0)$ and $\mu'(0)$.*

3.1.3 Light-Like Lines Correspond to Ideal Points

Hyperbolic space comes equipped with an ideal boundary, $\partial\mathcal{H}^n$, made up of points at infinity. We consider another model of hyperbolic n-space in order to deduce what kind of vector subspace corresponds to points in the infinite boundary of $\mathcal{H}^n$.

Hyperbolic n-space is also expressed through the ***Klein-Beltrami model of hyperbolic n-space***, $\mathbb{H}^n$, where hyperbolic n-space is identified with the unit ball B^n, and the ideal boundary of $\mathcal{H}^n$ is identified with the unit sphere $\mathbb{S}^{n-1}$. The isometry ψ takes the unit ball B^n to $\mathcal{H}^n$:

$$(x_1, \ldots, x_n) \mapsto \left(\frac{2x_1}{1-|x|^2}, \ldots, \frac{2x_n}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2} \right). \tag{3.5}$$

This map extends to take point $a = (a_1, \ldots, a_n)$ in $\mathbb{S}^{n-1}$ to the light-like line through 0 and $(a_1, \ldots, a_n, 1)$. Thus, points in the ideal boundary of hyperbolic n-space, or ***ideal points***, are represented by light-like lines, and every light-like line represents an ideal point.

The reader should note that there is a marked distinction between representing a point in $\mathcal{H}^n$ with a time-like *vector* in $\mathbb{R}^{n+1}$, and representing a point in $\partial\mathcal{H}^n$ with a light-like *line*, rather than a particular light-like vector within the line. Points in $\mathcal{H}^n$ can be represented with a specific time-like vector because of the relationship between hyperbolic distance of points and Lorentz inner product of time-like vectors. Picking positive time-like unit vectors yields a well-defined correspondence. However, for two sequences of points, $x_a(t)$ and $y_b(t)$ in $\mathcal{H}^n$, approaching points a and b in $\partial\mathcal{H}^n$ respectively, $d_{\mathcal{H}}(x_a(t), y_b(t)) \to \infty$ as $t \to \infty$, so a and b are infinitely far away from each other in the hyperbolic metric. Picking a specific vector to represent each of a and b within $\mathbb{R}^{n+1}$ would imply, by continuity, that there is a finite hyperbolic distance between them. This issue is resolved by representing ideal points a and b with light-like lines ℓ_a and ℓ_b, respectively.

One may naturally wonder what measurement between ideal points in $\partial\mathcal{H}^n$ is preserved if hyperbolic distance cannot be used. This is the motivation for looking at the rigidity of vectors and light-like lines under various conditions in chapter 2. Theorem 2.3.4 does not take on geometric meaning, as is, in $\mathcal{H}^n$. Further along in this section, we craft a hyperbolic invariant between ideal points and objects in $\mathcal{H}^n$. For an invariant of ideal points only, see the next chapter, where the Klein-Beltrami model of hyperbolic n-space is covered in more detail. We will use this model to talk about the role of vectors and lines in Lorentz Space play in generating the geometry of points and spheres in $\mathbb{S}^{n-1}$.

3.1.4 Space-like Vectors Correspond to Hyperbolic Hyperplanes

Points and lines in $\mathcal{H}^n \cup \partial\mathcal{H}^n$ discussed, and from this, we have found a hyperbolic geometric meaning for light-like lines and time-like vectors. Now, we generalize to higher-dimensional objects in $\mathcal{H}^n$, and it may come as no surprise that space-like vectors are involved in this last territory. In this way, every one-dimensional subspace of $\mathbb{R}^{n+1}$ corresponds to a geometric object in $\mathcal{H}^n$.

Definition 3.1.12. *A **hyperbolic** m**-plane** of $\mathcal{H}^n$ is the intersection of $\mathcal{H}^n$ with an $(m+1)$-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$. For a given hyperbolic m-plane $P = \mathcal{H}^n \cap V$ of $\mathcal{H}^n$, call the $(m+1)$-dimensional time-like subspace V the **time-like subspace supporting** P. A hyperbolic $(n-1)$-plane of $\mathcal{H}^n$ is called a **hyperplane of** $\mathcal{H}^n$.*

Let P be some m-dimensional hyperbolic plane in $\mathcal{H}^n$, and let V_P be the corresponding $(m+1)$-dimensional time-like subspace in $\mathbb{R}^{n+1}$. Recall from Chapter 2, by Lemma 2.1.10, for any $(m+1)$-dimensional time-like vector space V_P there is a space-like vector space W of dimension $n-m$ such that $W = (V_P)^L$. More specifically, we make the following observation.

Lemma 3.1.13. *The subspace V is n-dimensional and time-like if and only if V^L is 1-dimensional and space-like.*

Corollary 3.1.14. *(i) If P is a hyperplane in $\mathcal{H}^n$, and V_P is the supporting time-like subspace, then there is a unique positive space-like unit vector v such that $\langle v \rangle = V_P^L$. In this case, v is called **the positive unit vector Lorentz orthogonal to hyperplane** P.*

*(ii) If v is a space-like vector in $\mathbb{R}^{n+1}$, then there is a unique hyperplane $P = v^L \cap \mathcal{H}^n$ in $\mathcal{H}^n$ such that the supporting time-like subspace of P is Lorentz orthogonal to v. Hyperplane P is called **the hyperplane Lorentz orthogonal to** v.*

Hyperbolic Characterization of Two Linearly Independent Space-Like Vectors.

The above corollary yields a one-to-one correspondence between hyperplanes and positive space-like unit vectors, which we will study in the remainder of this section.

Definition 3.1.15. *Let P be a hyperplane of $\mathcal{H}^n$ and let $\lambda : \mathbb{R} \to \mathcal{H}^n$ be a geodesic line such that $\lambda(0)$ is in P. Then the hyperbolic line $L = \lambda(\mathbb{R})$ is said to be **Lorentz orthogonal** to P if P is the hyperplane of $\mathcal{H}^n$ Lorentz orthogonal to $\lambda'(0)$.*

The following gives a characterization for the interaction between two hyperplanes P and Q in $\mathcal{H}^n$ Lorentz orthogonal to two linearly independent space-like vectors v and w respectively. This characterization provides a starting point for how to think about the characterization of how n hyperplanes interact when they are Lorentz orthogonal to n linearly independent space-like vectors.

Theorem 3.1.16 ([19]). *Let v and w be linearly independent space-like vectors in $\mathbb{R}^{n+1}$. Then*

(i) vectors v and w span a space-like subspace V if and only if hyperplanes P and Q of $\mathcal{H}^n$, Lorentz orthogonal to v and w respectively, intersect nontrivially;

(ii) vectors v and w span a time-like subspace V if and only if hyperplanes P and Q of $\mathcal{H}^n$, Lorentz orthogonal to v and w respectively, are disjoint and have a common Lorentz orthogonal hyperbolic line.

(iii) vectors v and w span a light-like subspace V if and only if hyperplanes P and Q of $\mathcal{H}^n$, Lorentz orthogonal to v and w resp., meet at a point on the ideal boundary of $\mathcal{H}^n$, at infinity.

Hyperbolic Distance and Hyperbolic Angle Between Hyperplanes. We have covered the hyperbolic distance between points, and the hyperbolic angle between intersecting lines. This information is generalized to the hyperbolic distance between disjoint hyperplanes, and the hyperbolic angle between intersecting hyperplanes.

Theorem 3.1.17 ([19]). *Let v and w be space-like vectors in $\mathbb{R}^{n+1}$ that span a time-like vector subspace, and let P, Q be the hyperplanes of $\mathcal{H}^n$ Lorentz orthogonal to v, w, respectively. Then $\eta(v, w)$ is the hyperbolic distance from P to Q measured along the hyperbolic line N Lorentz orthogonal to P and Q. Moreover, $\langle v, w\rangle < 0$ if and only if v and w are oppositely oriented tangent vectors of N.*

By this theorem, and (2.6), we get for hyperplanes P and Q corresponding to positive space-like vectors v and w,

$$|\langle v, w\rangle| = ||v||||w|| \cosh d_{\mathcal{H}}(P, Q), \tag{3.6}$$

Note that positive space-like unit vectors v and w may still yield a negative Lorentz inner product, meaning they fit the description of acting as oppositely oriented tangent vectors of a hyperbolic line N. That is, we can synonymously think of the Lorentz time-like angle between space-like vectors v and w as the oriented hyperbolic distance between the hyperplanes Lorentz orthogonal to v and w.

Theorem 3.1.18 ([19]). *Let v and w be linearly independent space-like vectors in $\mathbb{R}^{n+1}$ such that the vector subspace V spanned by v and w is light-like. Then $\langle v, w\rangle < 0$ if and only if v and w are on opposite sides of the 1-dimensional light-like subspace of V.*

We recall theorem 2.1.16, and put this information together with the above theorem. For hyperplanes P and Q that meet at infinity, any two corresponding space-like vectors v and w Lorentz orthogonal to P and Q respectively, spanning a light-like subspace, satisfy that

$$|\langle v, w\rangle| = ||v||||w||, \tag{3.7}$$

and again, if positive space-like unit vectors are chosen to represent P and Q, it is always true that

$$|\langle v, w\rangle| = 1, \tag{3.8}$$

where $\langle v, w\rangle = 1$ if both vectors are on the same side of the light-like line in V, and $\langle v, w\rangle = -1$ if they lie on opposite sides.

Said another way, equation (3.8) can be rewritten as

$$\langle v, w\rangle = \cos\eta(v, w), \tag{3.9}$$

where θ equals 0 or π, dependent upon whether $\langle v, w\rangle$ equals 1 or -1 respectively. Generalizing from this idea, for positive space-like unit vectors v and w spanning a space-like subspace, we have seen that

$$\langle v, w\rangle = \cos\eta(v, w), \tag{3.10}$$

where $0 < \eta(v, w) < \pi$ is the Lorentz space-like angle between v and w. Let P and Q be the hyperplanes Lorentz orthogonal to v and w respectively.

Definition 3.1.19. *Let P and Q be two hyperplanes in $\mathcal{H}^n$ that are either intersecting in $\mathcal{H}^n$, or meet only at infinity. Let v and w be space-like unit vectors Lorentz orthogonal to P and Q respectively. The span of v and w is either time-like or light-like, respective to whether P and Q are intersecting in $\mathcal{H}^n$, or meeting only at infinity. The **hyperbolic angle between hyperplanes P and Q,** $\theta(P, Q)$ is $0 \le \theta(P, Q) = \eta(v, w) \le \pi$, where $\langle v, w\rangle = \cos\eta(v, w)$. The hyperbolic angle between P and Q is 0 when v and w are both positive or both negative and span a light-like subspace.*

Hyperbolic angle $\theta(P,Q) = \pi$ *when* v *and* w *span a light-like subspace and are opposite signs. In both these cases, hyperplanes* P *and* Q *meet only at infinity. Hyperbolic angle* $\theta(P,Q) = \pi/2$ *if and only if space-like vectors* v *and* w *are Lorentz orthogonal.*

Note that this definition is consistent with the hyperbolic angle between lines. Consider the case where lines *are* the hyperplanes, in $\mathcal{H}^2$. For two lines L_1 and L_2 in $\mathcal{H}^2$ intersecting at $L_1(0) = L_2(0)$, with respective Lorentz orthogonal positive space-like unit vectors v_1 and v_2, $\langle v_1, v_2\rangle = \langle L_1'(0), L_2'(0)\rangle$, so $\theta(L_1, L_2)$ is well-defined.

Theorem 3.1.20 ([19])**.** *Let* v *be a space-like vector and* w *a positive time-like vector in* $\mathbb{R}^{n+1}$*, and let* P *be the hyperplane of* $\mathcal{H}^n$ *Lorentz orthogonal to* v*. Then* $\eta(v,w)$ *is the hyperbolic distance from* $w/\big|||w|||\big|$ *to* P *measured along the hyperbolic line* N *passing through* $w/\big|||w|||\big|$ *Lorentz orthogonal to* P*. Moreover,* $\langle v, w\rangle < 0$ *if and only if* v *and* w *are on opposite sides of the hyperplane of* $\mathbb{R}^{n+1}$ *spanned by* P*.*

If w is a positive time-like unit vector in $\mathbb{R}^{n+1}$, and v a positive space-like unit vector with hyperplane P in $\mathcal{H}^n$ Lorentz orthogonal to v, then by Theorem 2.1.19,

$$\langle v, w\rangle = \sinh d_{\mathcal{H}}(P, w). \tag{3.11}$$

Theorem 3.1.16 describes the geometric correspondence to a 2-dimensional vector subspace spanned by two linearly independent space-like vectors. The next natural phase is to characterize the hyperbolic geometric properties for an n-dimensional subspace spanned by n linearly independent space-like vectors. This is broken into the three cases in which the subspace spanned is either space-like, time-like, or light-like.

Space-Like Vectors Spanning a Space-Like Subspace.

Lemma 3.1.21. *A collection of positive space-like unit vectors* $v_1, \ldots, v_n$ *in* $\mathbb{R}^{n+1}$ *spans an* n*-dimensional space-like subspace* V *if and only if there is a hyperbolic isometry* ϕ *taking the hyperplanes* $P_1, \ldots, P_n$ *in* $\mathcal{H}^n$*, respectively Lorentz orthogonal to* $v_1, \ldots, v_n$*, to hyperplanes* $\Pi_1, \ldots, \Pi_n$*, where* $(0, \ldots, 0, 1)$ *is the only point common to all* Π_i*, for* $i = 1, \ldots, n$*.*

Proof. Let $v_1, \ldots, v_n$ be a collection of positive space-like unit vectors in $\mathbb{R}^{n+1}$, and let $P_1, \ldots P_n$ be the hyperplanes Lorentz orthogonal to $v_1, \ldots, v_n$, respectively.

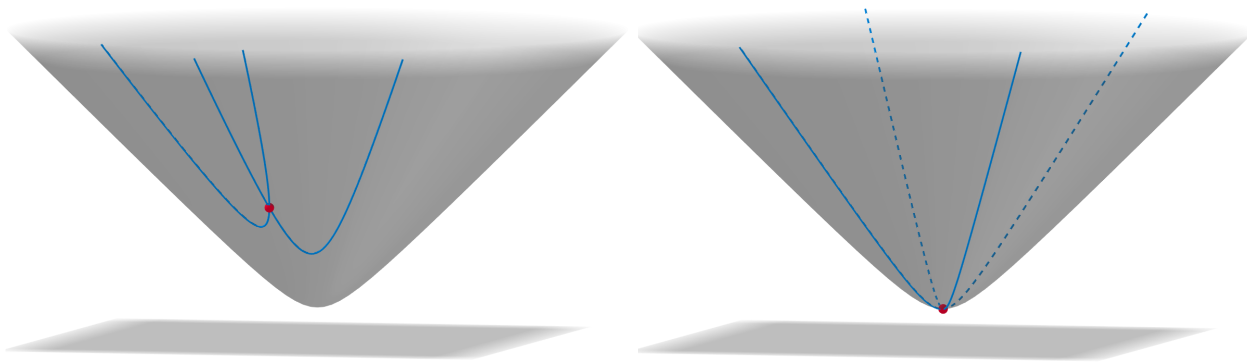

Figure 3.3: An example of hyperplanes Lorentz orthogonal to positive space-like unit vectors that span a space-like subspace. In $\mathbb{R}^3$, $n = 2$. A collection of n hyperplanes, in blue, Lorentz orthogonal to space-like vectors spanning an n-dimensional space-like subspace (left), must be isometric to a collection of n hyperplanes only meeting at the red point, $(0, \ldots, 0, 1)$ (right).

Assume $v_1, \ldots, v_n$ form a basis for a space-like subspace V. Then since $PO(n, 1)$ acts transitively on m-dimensional space-like subspaces (theorem 2.2.2), there is a positive Lorentz transformation ϕ that takes V to space-like subspace $\mathbb{R}^n = \{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1} : x_i \in \mathbb{R}\}$, and $v_1, \ldots, v_n$ to a basis $\phi(v_1), \ldots, \phi(v_n)$of $\mathbb{R}^n$. For each i, Lorentz transformation ϕ takes the n-dimensional time-like subspace V_i supporting P_i to the n-dimensional time-like subspace $\phi(V_i)$ that is Lorentz orthogonal to $\phi(v_i)$, and ϕ restricts to an isometry $\Phi = \phi|_{\mathcal{H}^n}$ of $\mathcal{H}^n$. So for each i, $\Phi(P_i) = \phi(V_i) \cap \mathcal{H}^n$ is a hyperplane in $\mathcal{H}^n$ Lorentz orthogonal to $\phi(v_i)$. Since each $\phi(v_i)$ is in $\mathbb{R}^n$, hyperplane $\Phi(P_i)$ must necessarily contain point $(0, \ldots, 0, 1)$, and furthermore, this is the only point common to all $\Phi(P_i)$.

Now assume there is a hyperbolic isometry Φ of $\mathcal{H}^n$ such that, for each $i = 1, \ldots, n$, we get $\Phi(P_i) = \Pi_i$, where each Π_i is a hyperplane of $\mathcal{H}^n$ containing point $(0, \ldots, 0, 1)$, and $(0, \ldots, 0, 1)$ is the only point common to all Π_i for $i = 1, \ldots, n$. Isometry Φ extends to a positive Lorentz transformation of $\mathbb{R}^{n+1}$. Each hyperplane $\Pi_i = \Phi(P_i)$ is supported by n-dimensional time-like subspace $\Phi(V_i)$, where V_i supports P_i, and $\Phi(V_i)$ is Lorentz orthogonal to a space-like unit vector $\Phi(v_i)$. Since Π_i contains $(0, \ldots, 0, 1)$, vector $\Phi(v_i)$ is in $\mathbb{R}^n = \{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1} : x_i \in \mathbb{R}\}$. Call the time-like vector through $(0, \ldots, 0, 1)$ by v_0. Since $\bigcap_i \Phi(V_i) = \langle v_0 \rangle$, consequently, $\Phi(v_1) + \ldots + \Phi(v_n) = \text{span}\{\Phi(v_1), \ldots, \Phi(v_n)\} = \mathbb{R}^n$. Thus, $v_1, \ldots, v_n$ span an n-dimensional space-like subspace. $\square$

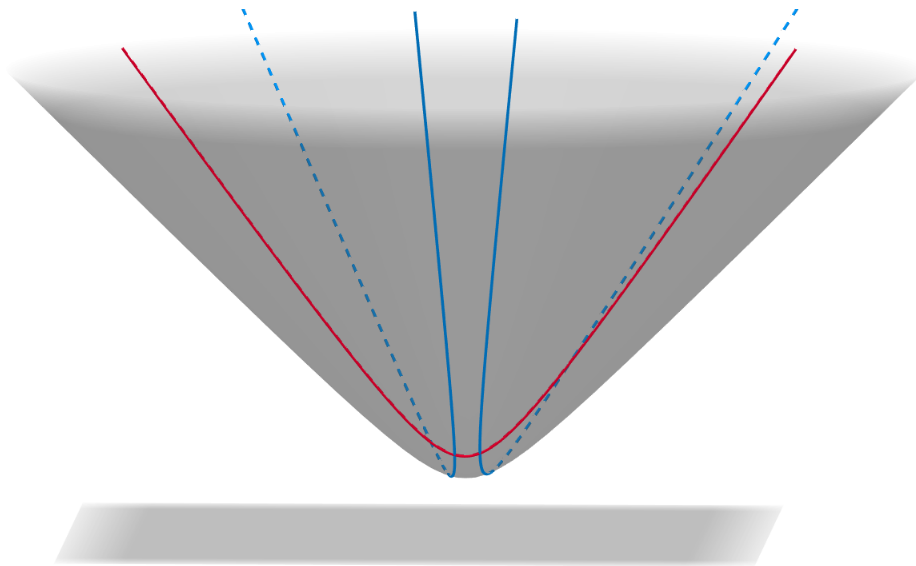

Figure 3.4: A collection of n hyperplanes, in blue, Lorentz orthogonal to space-like vectors spanning an n-dimensional time-like subspace. This collection is characterized by being commonly Lorentz orthogonal to a unique hyperbolic line, in red. In the $n = 2$ case, the hyperplanes are also lines.

Any two hyperplanes P_i and P_j Lorentz orthogonal to space-like unit vectors v_i and v_j in the span of an n-dimensional space-like subspace must intersect in $\mathcal{H}^n$. This is a characteristic that is unique to a collection of space-like unit vectors spanning a space-like subspace.

Space-Like Vectors Spanning a Time-Like Subspace.

Lemma 3.1.22. *A collection of positive space-like unit vectors $v_1, \ldots, v_n$ in $\mathbb{R}^{n+1}$ spans an n-dimensional time-like subspace V if and only if for the hyperplanes $P_1, \ldots, P_n$ Lorentz orthogonal to $v_1, \ldots, v_n$, respectively, there is a unique hyperplane Q supported by V such that the hyperbolic angle $\theta(P_i, Q)$ for every i is $\pi/2$.*

Proof. Let $v_1, \ldots, v_n$ be a collection of positive space-like unit vectors in $\mathbb{R}^{n+1}$, and let $P_1, \ldots, P_n$ be the hyperplanes Lorentz orthogonal to $v_1, \ldots, v_n$, respectively.

Assume $v_1, \ldots, v_n$ are a basis for a time-like subspace V. Then since V is time-like and n-dimensional, $Q = V \cap \mathcal{H}^n$ is a hyperplane in $\mathcal{H}^n$. Additionally, since V is n-dimensional and each $\langle v_i \rangle^L$ is n-dimensional, the intersection $V \cap \langle v_i \rangle^L$ is an $(n-1)$-dimensional subspace. The Lorentz complement V^L is a one-dimensional and space-like subspace. Denote the positive unit vector in V^L as v. Then $\langle v, v_i \rangle = 0$. Thus, the subspace $\operatorname{span}\{v, v_i\}$ must be space-like, because

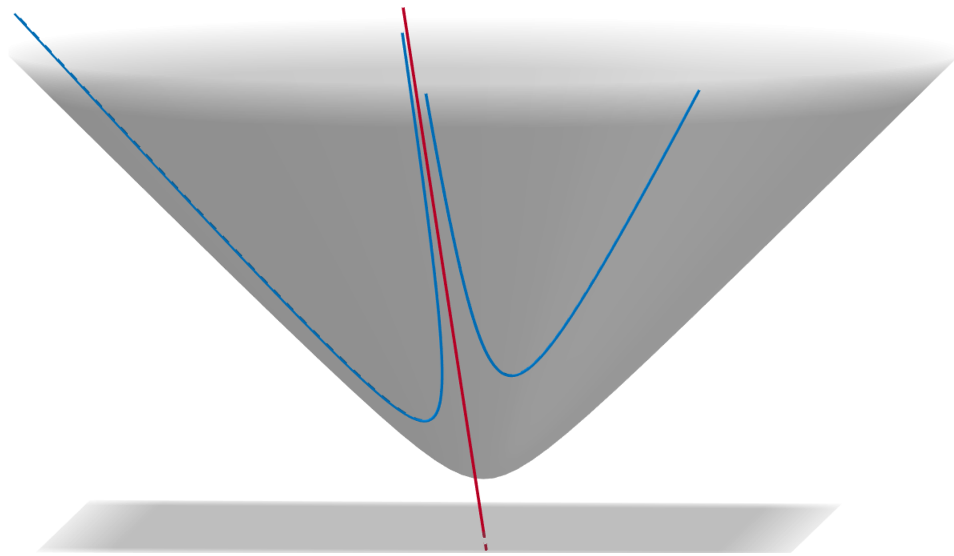

Figure 3.5: A collection of hyperplanes, in blue, Lorentz orthogonal to space-like vectors spanning an n-dimensional light-like subspace. All hyperplanes meet at a point in the ideal boundary of hyperbolic space, picked out by the light-like line, in red.

$|\langle v, v_i \rangle| < ||v|| ||w|| = 1$. By theorem 3.1.16, Q intersects each P_i in $\mathcal{H}^n$. Since, for each i, $\langle v_i, v \rangle = 0 = \cos \eta(v_i, v_Q)$, we get that $\theta(P_i, Q) = \pi/2$ for each i.

Assume there is one and only one hyperplane Q such that $\theta(P_i, Q) = \pi/2$ for every i. Then Q is the intersection of an n-dimensional time-like subspace V_Q with $\mathcal{H}^n$ that is Lorentz orthogonal to a positive space-like unit vector v_Q, so that v_Q is the only positive space-like unit vector such that $\langle v_Q, v_i \rangle = 0$ for every i. This means that $W = \text{span}\{v_1, \ldots v_n\}$ must be time-like, and that $W \subset V_Q$. Moreover, V_Q is the only n-dimensional time-like subspace that W is contained in. If $\dim W < \dim V_Q$, this would not hold. Thus, $V_Q = \text{span}\{v_1, \ldots, v_n\}$. □

Any two hyperplanes P_i and P_j Lorentz orthogonal to space-like vectors v_i, v_j in the span of V, respectively, may be either disjoint, intersect, or meet only at infinity.

Space-Like Vectors Spanning a Light-Like Subspace.

Lemma 3.1.23. *Let V be an n-dimensional light-like subspace in $\mathbb{R}^{n+1}$ spanned by n linearly independent space-like vectors $v_1, \ldots, v_n$. Then there is no pair of v_i, v_j in $v_1, \ldots, v_n$ that spans a time-like subspace.*

Proof. Let V be an n-dimensional light-like subspace in $\mathbb{R}^{n+1}$. Let $v_1, \ldots, v_n$ be a collection of linearly independent space-like vectors such that $\text{span}\{v_1, \ldots, v_n\} = V$. Take any distinct v_i and v_j within $v_1, \ldots, v_n$ and let $W = \text{span}\{v_i, v_j\}$. Then W cannot be time-like because if it was, then W would contain a time-like vector by definition, which would mean V contains a time-like vector. This violates the definition of a light-like subspace. □

It is possible for a light-like vector space to contain a space-like subspace. Build one such example by taking an $(n-1)$-dimensional space-like subspace W, spanned by linearly independent space-like vectors $v_1, \ldots, v_{n-1}$. Pick space-like vector w such that $\text{span}\{v_1, w\}$ is light-like. Then $\text{span}\{w, v_1, \ldots, v_{n-1}\} = W$ is an n-dimensional light-like subspace.

Lemma 3.1.24. *A collection of positive space-like unit vectors $v_1, \ldots, v_n$ in $\mathbb{R}^{n+1}$ spans an n-dimensional light-like subspace V if and only if the corresponding hyperplanes $P_1, \ldots, P_n$ Lorentz orthogonal to $v_1, \ldots, v_n$ respectively, all meet at a unique point at infinity, and there is not a hyperplane P_k distinct from P_i for every $i = 1, \ldots, n$, that intersects every P_i at a hyperbolic angle of $\pi/2$.*

Proof. Let $v_1, \ldots, v_n$ be a collection of linearly independent positive space-like unit vectors in $\mathbb{R}^{n+1}$ and let $P_1, \ldots, P_n$ be the hyperplanes in $\mathcal{H}^n$ Lorentz orthogonal to $v_1, \ldots, v_n$, respectively.

Assume $v_1, \ldots, v_n$ are a basis for a light-like subspace V. Since V is light-like and n-dimensional, there is only one light-light line ℓ in V, and $\ell = V^L$. For each $i = 1, \ldots, n$, let W_i be the n-dimensional time-like Lorentz complement of v_i in $\mathbb{R}^{n+1}$ supporting P_i. Since $\langle v_i \rangle \subset V$, we obtain that $\ell = V^L \subset \langle v_i \rangle^L = W_i$. Line ℓ represents a point x at infinity, in $\partial\mathcal{H}^n$, so hyperplane $W_i \cap \mathcal{H}^n = P_i$ meets x at infinity. There is no other point y at infinity common to every P_i, else V is not light-like. Since V is not time-like, there is no hyperplane P_k intersecting every P_i at a hyperbolic angle of $\pi/2$.

On the other hand, assume hyperplanes $P_1, \ldots, P_n$ meet at a unique point at infinity, x, and that there is no hyperplane that intersects every hyperplane P_i at hyperbolic angle $\pi/2$, for $i = 1, \ldots, n$. Let $\langle x \rangle$ be the line through point x. Since every P_i meets x at infinity, the time-like n-dimensional subspace W_i supporting P_i contains $\langle x \rangle$. Thus, the positive space-like unit vector v_i Lorentz orthogonal to W_i is contained in the n-dimensional light-like subspace $V = \langle x \rangle^L$ for each $i = 1, \ldots, n$, and $W = \text{span}\{v_1, \ldots, v_n\}$ is a light-like subspace contained in V. Assume that the dimension of W is m, where $1 \leq m < n$. Then $\dim(V \cap W^L) \geq 2$, else $V \cap W^L = \langle x \rangle$, meaning that $\langle x \rangle + W = V$, which would imply $W = V$. Since W^L and V are light-like and both share $\langle x \rangle$, $V \cap W^L$ is light-like. Let v_k be a positive space-like unit vector in $V \cap W^L$. Let W_k be the time-like subspace such that $W_k = \langle v_k \rangle^L$. Then W_k intersects $\mathcal{H}^n$ as a hyperplane that meets x at infinity, and intersects every P_i at a hyperbolic angle of $\pi/2$, which is a contradictions. Thus, $W = \text{span}\{v_1, \ldots, v_n\} = V$. □

3.1.5 Hyperbolic Ratio

Because the hyperbolic distance between a hyperplane or point in $\mathcal{H}^n$, and a point in the ideal boundary is infinite, a different isometric invariant is needed as a measurement between these objects.

Definition 3.1.25. *Let p_1 and p_2 both be fixed points, or both be fixed hyperplanes in $\mathcal{H}^n$, and let a be an ideal point. Let p_t be a sequence of all points if p_1 and p_2 are points, or all hyperplanes if p_1 and p_2 are hyperplanes, in $\mathcal{H}^n$ converging towards a, that is, $p_t \to a$ as $t \to \infty$. Then*

$$(a, p_1, p_2) = \lim_{t\to\infty} \frac{\cosh d_{\mathcal{H}}(p_t, p_1)}{\cosh d_{\mathcal{H}}(p_t, p_2)} \tag{3.12}$$

is the ***hyperbolic ratio of ideal point*** *a* ***with*** *p_1* ***and*** *p_2.*

Lemma 3.1.26. *Let p_1 and p_2 both be fixed points, or both be fixed hyperplanes in $\mathcal{H}^n$, and let a be an ideal point of $\mathcal{H}^n$. Then the hyperbolic ratio (a, p_1, p_2) always exists, and*

$$(a, p_1, p_2) = (\ell_a, v_{p_1}, v_{p_2}), \tag{3.13}$$

where ℓ_a is the light-like line through point a, and where v_{p_1} and v_{p_2} are the positive unit vectors corresponding to p_1 and p_2.

Proof. Let v_{p_1} and v_{p_2} be positive unit vectors, respectively corresponding to p_1 and p_2 in $\mathcal{H}^n$, both time-like or both space-like dependent up whether each of p_1 and p_2 are both hyperplanes or both points. Let p_t be a sequence of all hyperplanes if p_1 and p_2 are hyperplanes, or all points if p_1 and p_2 are points, where $p_t \to a$ as $t \to \infty$. Let v_t be the corresponding sequence of all space-like or all time-like positive unit vectors, respectively. Then

$$(a, p_1, p_2) = \lim_{t\to\infty} \frac{\cosh d_{\mathcal{H}}(p_t, p_1)}{\cosh d_{\mathcal{H}}(p_t, p_2)} = \frac{\langle v_t, v_{p_1} \rangle}{\langle v_t, v_{p_2} \rangle} = (\ell_a, v_{p_1}, v_{p_2}) = \frac{\langle v_a, v_{p_1} \rangle}{\langle v_a, v_{p_2} \rangle}, \tag{3.14}$$

where v_a is any vector in the light-like line ℓ_a corresponding to ideal point a. □

The hyperbolic ratio is always positive, making it a restriction of the concept of the Lorentz ratio.

3.1.6 Basis Vectors Characterization

The rigidity theorems in Chapter 2 translate into rigidity theorems for objects in hyperbolic Space and, as will be seen in Chapter 4, objects in the ideal boundary. Each of the rigidty theorems in Chapter 2 used an independent subcollection in $\mathbb{R}^{n+1}$, so now we translate what the corresponding collections of hyperplanes and points in $\mathcal{H}^n$ look like in accordance with this. First, if a basis for $\mathbb{R}^{n+1}$ is made up of only positive space-like unit vectors, this corresponds to a collection of hyperplanes with the following characterization.

Lemma 3.1.27. *Let $\{v_1, \ldots, v_{n+1}\}$ be a collection of positive space-like unit vectors in $\mathbb{R}^{n+1}$ where $\{P_1, \ldots, P_{n+1}\}$ are the collection of $n+1$ hyperplanes in $\mathcal{H}^n$, where v_i is Lorentz orthogonal to P_i for each $1 \le i \le n+1$. Then $\{v_i\}$ is a basis for $\mathbb{R}^{n+1}$ if and only if the hyperplanes:*

1. *do not all meet a common unique point at infinity,*
2. *do not all commonly intersect a unique hyperplane at hyperbolic angle $\pi/2$, and*
3. *are not isometric to a collection of hyperplanes $\Pi_1, \ldots, \Pi_{n+1}$ that contain one common point $(0, \ldots, 0, 1)$ in $\mathcal{H}^n$.*

This theorem is a conclusion drawn from the three main statements of the previous section.

Definition 3.1.28. *Let $X_k = \{p_1, \ldots, p_k\}$ be a collection of k points in $\mathcal{H}^n$, where $2 \le k \le n+1$. Collection X_k is an **independent collection of k points in** $\mathcal{H}^n$ if $p_1, \ldots, p_k$ do not all lie in a common $(k-2)$-plane in $\mathcal{H}^n$. Here 1-plane refers to a line in $\mathcal{H}^n$, and 0-plane refers to a point in $\mathcal{H}^n$.*

Note that if X_k is an independent collection of points in $\mathcal{H}^n$, then subcollection X_m of X_k, where $2 \le m < k$ is also automatically an independent collection of m points in $\mathcal{H}^n$.

Lemma 3.1.29. *Let $\{p_1, \ldots, p_k\}$ be a collection of k points in $\mathcal{H}^n$, where $2 \le k \le n+1$, and let $\{v_1, \ldots, v_k\}$ be a collection of positive time-like unit vectors in $\mathbb{R}^{n+1}$. Then $\{p_i\}$ is an independent collection of k points in $\mathcal{H}^n$ if and only if $\{v_i\}$ is a linearly independent collection of vectors in $\mathbb{R}^{n+1}$.*

Note that if $\{p_1, \ldots, p_k\}$ is independent, then no three points p_i, p_j, p_l in the collection are collinear.

If a basis for $\mathbb{R}^{n+1}$ is composed of all positive time-like unit vectors, then there is a corresponding collection of points in $\mathcal{H}^n$ with the following criteria.

Corollary 3.1.30. *A collection of $n+1$ points in $\mathcal{H}^n$, $\{p_1, \ldots, p_{n+1}\}$ is independent if and only if the corresponding collection of positive time-like unit vectors $\{v_1, \ldots, v_{n+1}\}$ is a basis for $\mathbb{R}^{n+1}$.*

3.2 Rigidity of Hyperbolic Points, $(n-1)$-Planes, and Ideal Points

We now have everything needed to translate theorem 2.3.12 into a statement regarding the rigidity of intermingled collections of points, ideal points, and hyperplanes in $\mathcal{H}^n$. This theorem accomplishes three things: it reduces the amount of conformal invariant information used between hyperplanes from the statements made in [13], it brings the statements made in [13] into the setting of hyperbolic n-space, where hyperbolic points are able to be considered additionally, and it uses a new conformal invariant (the hyperbolic ratio) so that the rigidity of intermingled collections can be handled where they previously could not. The theorem is stated and proved below. Theorem 2.3.8 will be interpreted in Chapter 4, separately; the Lorentz invariant used is best interpreted within the context of conformal geometry of $\mathbb{S}^{n-1}$.

For the following theorem, $\eta(p_i, p_j)$ and $\eta(p'_i, p'_j)$ is used to denote hyperbolic distance $d_{\mathcal{H}}(p_i, p_j) = d_{\mathcal{H}}(p'_i, p'_j)$, when p_i, p_j and p'_i, p'_j are either pairs of points or pairs of disjoint hyperplanes in $\mathcal{H}^n$, and $\eta(p_i, p_j) = \eta(p'_i, p'_j)$ denotes equal hyperbolic angles when p_i, p_j and p'_i, p'_j are intersecting hyperplanes in $\mathcal{H}^n$.

Theorem 3.2.1. *Let $\mathcal{C} = \{p_\alpha, p_\beta, p_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ and $\mathcal{C}' = \{p'_\alpha, p'_\beta, p'_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ be two collections where p_α, p'_α are points in $\mathcal{H}^n$, p_β, p'_β are hyperplanes in $\mathcal{H}^n$, and p_γ, p'_γ are points at infinity in $\partial\mathcal{H}^n$, where no p_γ, p'_γ respectively lies on the boundary of any p_β, p'_β. Let $\mathcal{C}$ and $\mathcal{C}'$ contain corresponding subcollections of either $n+1$ hyperplanes or $n+1$ points, $\{p_i\}$ and $\{p'_i\}$ in $\mathcal{H}^n$, such that $\eta(p_i, p_j) = \eta(p'_i, p'_j)$ for each distinct pair $1 \leq i, j \leq n+1$. Further, assume:*

(i) If $\{p_i\}$ and $\{p'_i\}$ are hyperplanes in $\mathcal{H}^n$, then $\{p_i\}$ and $\{p'_i\}$, respectively:

(a) do not all meet a common unique point at infinity,

(b) do not all commonly intersect a unique hyperplane at hyperbolic angle $\pi/2$, and

(c) are not isometric to a collection of hyperplanes $\Pi_1, \ldots, \Pi_{n+1}$ that contain one common point $(0, \ldots, 0, 1)$ in $\mathcal{H}^n$;

(ii) If $\{p_i\}$ and $\{p'_i\}$ are points in $\mathcal{H}^n$, then $\{p_i\}$ and $\{p'_i\}$ are each independent subcollections of $n+1$ points in $\mathcal{H}^n$.

Then

$$\eta(p_\alpha, p_i) = \eta(p'_\alpha, p'_i), \tag{3.15}$$

$$\eta(p_\beta, p_i) = \eta(p'_\beta, p'_i), \tag{3.16}$$

for each i, for all points p_α, p'_α, for all hyperplanes p_β, p'_β in $\mathcal{H}^n$, and $(p_\gamma, p_i, p_j) = (p'_\gamma, p'_i, p'_j)$ for each distinct pair i, j in the independent subcollection index, and all ideal points p_γ, p'_γ if and only if there is a unique hyperbolic isometry ϕ belonging to $I(\mathcal{H}^n)$ such that $\phi(p_\alpha) = p'_\alpha$, $\phi(p_\beta) = p'_\beta$, and $\phi(p_\gamma) = p'_\gamma$ for all α, β, γ in $\mathcal{A}$

Proof. Assume $\mathcal{C}$ and $\mathcal{C}'$ are two collections of points p_α, p'_α in $\mathcal{H}^n$, hyperplanes p_β, p'_β in $\mathcal{H}^n$, and ideal points p_γ, p'_γ in $\partial\mathcal{H}^n$ with subcollection of $n+1$ point or $n+1$ hyperplanes $\{p_i\}$ and $\{p'_i\}$ fitting the description above. Then $\mathcal{C}$ and $\mathcal{C}'$ correspond to collections $\mathcal{V} = \{v_\alpha, v_\beta, v_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ and $\mathcal{V}' = \{v'_\alpha, v'_\beta, v'_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$, respectively, of positive space-like unit vectors v_α, v'_α Lorentz orthogonal to p_α, p'_α respectively, positive time-like unit vectors v_β, v'_β through points p_β, p'_β, resp., and light-like lines $\ell_\gamma, \ell'_\gamma$ in $\mathbb{R}^{n+1}$ through ideal point p_γ, p'_γ, resp. Each of $\mathcal{V}$ and $\mathcal{V}'$ have subcollections $\{v_i\}$ and $\{v'_i\}$ of $n+1$ space-like vectors, if $\{p_i\}$ and $\{p'_i\}$ resp. are hyperplanes, or $n+1$ time-like vectors if $\{p_i\}$ and $\{p'_i\}$ are points in $\mathcal{H}^n$. These subcollections $\{v_i\}$ and $\{v'_i\}$ are both a basis of vectors for $\mathbb{R}^{n+1}$ by Lemmas 3.1.27 and 3.1.29. Moreover, our set up gives us that $\langle p_\alpha, p_i \rangle = \langle p'_\alpha, p'_i \rangle$, $\langle p_\beta, p_i \rangle = \langle p'_\beta, p'_i \rangle$, and $(p_\gamma, p_i, p_j) = (p'_\gamma, p'_i, p'_j)$. Thus, by theorem 2.3.12, and since all vectors are positive, there is a unique positive Lorentz transformation Φ such that $\Phi(v_\alpha) = v'_\alpha$, $\Phi(v_\beta) = v'_\beta$ and $\Phi(v_\gamma) = v'_\gamma$. Lorentz transformation Φ restricts to a unique isometry, ϕ, of $\mathcal{H}^n$. Thus, we get that $\phi(p_\alpha) = p'_\alpha$, $\phi(p_\beta) = p'_\beta$, and $\phi(p_\gamma) = p'_\gamma$, for all α, β, γ in $\mathcal{A}$. □

This rigidity result yields a statement for collections of all hyperbolic points, all hyperplanes, intermingled hyperbolic points, hyperplanes, and ideal points. It does not say what to do if your collections are entirely composed of ideal points. Moreover, there is another angle to explore this statement from that has not yet been given attention. [13] state their results in the language of the geometry of balls and points in $\mathbb{S}^{n-1}$, as their motivation was to generalize the work of [3]. One motivation of this dissertation is to build upon the work of [3] and [13] as well, so we now turn our attention to this context in the next chapter, as well as explore other rigidity questions within the geometry of circles.

CHAPTER 4

RIGIDITY OF SPHERES AND POINTS IN $\mathbb{S}^{N-1}$

In this chapter, we begin by building a dictionary, this time between the language of Lorentz space and that of the geometry of spheres. hyperbolic space bridges the gap between the two, so much of the lexicon translates directly from Chapter 3. In this chapter, however, we pay closer attention to orientation-preserving and orientation-reversing transformations, and develop rigidity statements that take both kinds of transformations into consideration. Here, the inversive ratio between again translates to an invariant of the geometry, here called the *inversive ratio* of a point and two circles. Moreover, the geometry of circles behaves surprisingly similarly to the geometry of points in Euclidean space, so we develop a notion of a *circle-line* and *circle-plane* in order to draw a correspondence between a collection of linearly independent vectors in Lorentz space and what is deemed an *independent* collection of circles. We end by turning our attention to the rigidity of general inversive distance circle packings. First, these circle configurations are discussed in the context of convex circle-polyhedra, where a Cauchy-style rigidity theorem is presented. As was discussed in the introduction, without the requirement of convexity, inversive distance circle packings are not globally rigid, in general. The last topic of this dissertation is a look into how much extra inversive distance information is sufficient for the global rigidity of general inversive distance circle packings. Such a statement can have practical applications when convexity is not guaranteed.

4.1 The Geometry of Spheres and Points in $\hat{\mathbb{R}}^{n-1}$

The first thing to point out is that this is conformal geometry, and while it is developed in $\hat{\mathbb{R}}^{n-1}$, this information can be transferred to the setting of $\mathbb{S}^{n-1}$ via stereographic projection. The stereographic projection map is conformal, so all information concerning the invariants and tranformations is preserved. Within this section, the geometry of spheres and points is developed intrinsically. In the next section, we lay out the correspondence between the geometry developed here, and the geometry of Lorentz space.

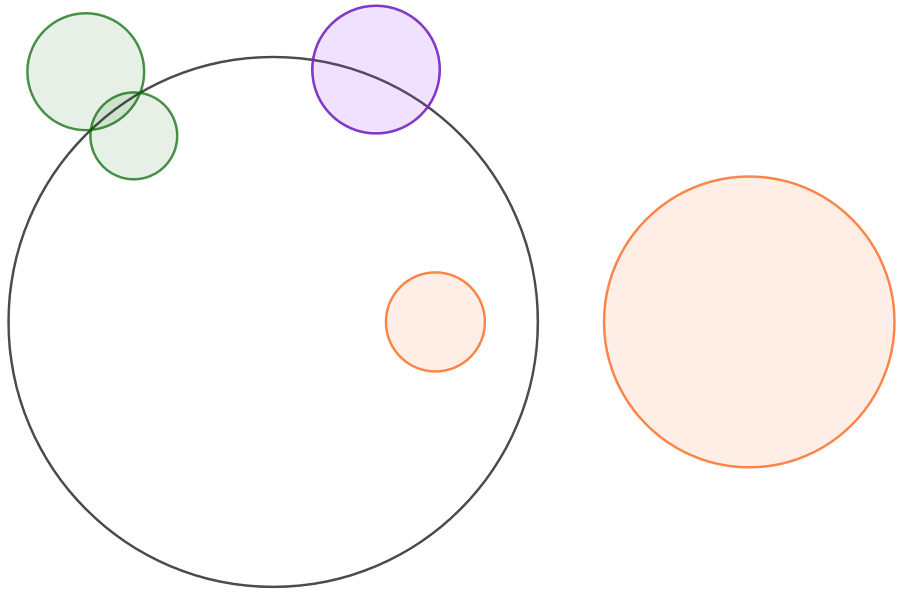

Figure 4.1: Three examples of an inversion through a circle. Consider the inversion fixing the black circle. This inversion takes each green circle (and disk) to the other and each orange circle to the other. Since the purple circle is orthogonal to the black circle, it is taken to itself. Each circle's orientation is reversed.

4.1.1 Möbius Transformations and Inversive Transformations of $\hat{\mathbb{R}}^{n-1}$

The reference for this section is [2]. A ***sphere*** in $\hat{\mathbb{R}}^{n-1}$ will refer to an $(n-2)$-sphere,

$$S = S(a, r) = \{x \in \mathbb{R}^{n-1} : |x - a| = r\}, \tag{4.1}$$

where $a \in \mathbb{R}^{n-1}$, and $r > 0$.

A ***reflection (or inversion)*** through $S(a, r)$ is the function ϕ defined by

$$\phi(x) = a + \left(\frac{r}{|x - a|}\right)^2 (x - a). \tag{4.2}$$

When $S(0, 1) = \mathbb{S}^{n-2}$, this is

$$\phi(x) = \frac{x}{|x|^2}. \tag{4.3}$$

Definition 4.1.1. *An **inversive transformation** acting in $\hat{\mathbb{R}}^{n-1}$ is a finite composition of reflections through spheres.*

Definition 4.1.2. *The group of Inversive transformations acting in $\hat{\mathbb{R}}^{n-1}$ is called the **Inversive group**, and is denoted by* $\mathrm{Inv}(\hat{\mathbb{R}}^{n-1})$.

Theorem 4.1.3 ([2]). *Every reflection is orientation-reversing and conformal.*

Corollary 4.1.4 ([2]). *A composition of an even number of reflections is orientation-preserving. A composition of an odd number of reflections is orientation-reversing.*

Definition 4.1.5. *A **Möbius transformation** acting in* $\hat{\mathbb{R}}^{n-1}$ *is an inversive transformation that is orienation-preserving.*

Note that any Möbius transformation is then a composition of an even number of reflections.

Definition 4.1.6. *The **Möbius group*** $\text{Möb}(\hat{\mathbb{R}}^{n-1})$ *acting in* $\hat{\mathbb{R}}^{n-1}$ *is the subgroup of* $\text{Inv}(\hat{\mathbb{R}}^{n-1})$ *consisting of all Möbius transformations in* $\text{Inv}(\hat{\mathbb{R}}^{n-1})$*.*

The subgroup $\text{Möb}(\hat{\mathbb{R}}^{n-1})$ is an index 2 subgroup of $\text{Inv}(\hat{\mathbb{R}}^{n-1})$ since every inversive transformation is either orientation-preserving or orientation-reversing.

Theorem 4.1.7 ([2]). *Let* ϕ *be any inversive transformation, and let* Σ *be any sphere in* $\hat{\mathbb{R}}^{n-1}$*. Then* $\phi(\Sigma)$ *is also a sphere.*

Theorem 4.1.8 ([2]). *Let* Σ *be any sphere,* σ *the reflection in* Σ*, and* I *the identity map. If* ϕ *is any inversive transformation which fixes each* x *in* Σ*, then either* $\phi = I$ *or* $\phi = \sigma$*.*

Corollary 4.1.9 ([2]). *Any two reflections are conjugate in* $\text{Inv}(\hat{\mathbb{R}}^{n-1})$*.*

Theorem 4.1.10 ([2]). $\text{Inv}(\hat{\mathbb{R}}^{n-1})$ *with the topology of uniform convergence in the chordal metric is isomorphic as a topological group to the group* $O^+(n, 1)$*.*

4.1.2 Absolute Cross Ratio of Points in $\hat{\mathbb{R}}^{n-1}$

Definition 4.1.11. *Given four distinct points* x, y, u, v *in* $\hat{\mathbb{R}}^{n-1}$*, the **absolute cross-ratio** of these points is*

$$|x, y, u, v| = \frac{d(x, u)d(y, v)}{d(x, y)d(u, v)} = \frac{|x - u| \cdot |y - v|}{|x - y| \cdot |u - v|}, \tag{4.4}$$

where d *is the chordal metric on* $\hat{\mathbb{R}}^{n-1}$*.*

Note that changing the order of x, y, u, and v will change the value of the absolute cross ratio, so this value is considered up to ordered 4-tuples.

Theorem 4.1.12 ([2]). *A map* $\phi : \hat{\mathbb{R}}^{n-1} \to \hat{\mathbb{R}}^{n-1}$ *is an inversive transformation if and only if it preserves absolute cross-ratios.*

4.1.3 The Geometry of Circles and Points in $\mathbb{S}^2$

We now lay out the intrinsic geometry of circles in $\mathbb{S}^2$. Note that all the information in the section prior still holds.

Theorem 4.1.13 ([2])**.** *$\text{Möb}(\mathbb{S}^2)$ is isomorphic to $\text{SL}(2,\mathbb{C})/\{\pm\lambda I\} = \text{PSL}(2,\mathbb{C})$, where $\lambda \in \mathbb{R}$.*

Cross Ratio. A ***circle*** in $\mathbb{S}^2$ is the stereographic projection of a 1-sphere in $\hat{\mathbb{R}}^2$ onto $\mathbb{S}^2$. When considering objects in the 2-sphere, one advantage is one can move seamlessly between the equivalent spaces $\mathbb{S}^2$, $\mathbb{C}$, and $\hat{\mathbb{R}}^2$. We do this now to define the cross ratio of 4 points.

Definition 4.1.14. *Let x, y, u, v be points in $\mathbb{S}^2 = \hat{\mathbb{C}}$. Then the **cross ratio** of these points is*

$$[x, y, u, v] = \frac{(x-u)(y-v)}{(x-y)(u-v)}, \tag{4.5}$$

and $[x, y, u, v]$ is a real number when x, y, u, v are points lying on a circle in $\mathbb{S}^2$.

Inversive Distance. It's been established that for a given circle C, there are two open disks it bounds and reflection I_C fixing C either fixes the two disks (ie, I_C is the identity), or swaps the two disks. Because of this, we wish to develop a conformal invariant that keeps track of both angle and orientation. The following definitions are referenced from [7], where the details are worked out thoroughly.

Definition 4.1.15. *A circle C is **oriented** in $\mathbb{S}^2$ if it is the boundary of the unique open disk D that lies to the left of C as one travels in the direction of the orientation of C.*

Any circle C bounds two open disks, one called the ***interior disk***, D, and the other called the ***exterior disk***. An orientation must be established to distinguish one from the other; the interior disk lies to the left as one travels in the direction of the orientation of C, and the exterior disk lies to the right. In this chapter, unless otherwise specified, it is assumed that a circle C is oriented, so that a given circle C comes equipped with interior disk D without explicitly mentioning it. If only the boundary circle is being used, it will be described as an *unoriented circle.*

Now that orientation has been established, we develop the ***inversive distance*** between a pair of circles. This is a real number assigned to a pair of circles that measures the interaction between the pair. It is a conformal invariant and keeps track of orientation.

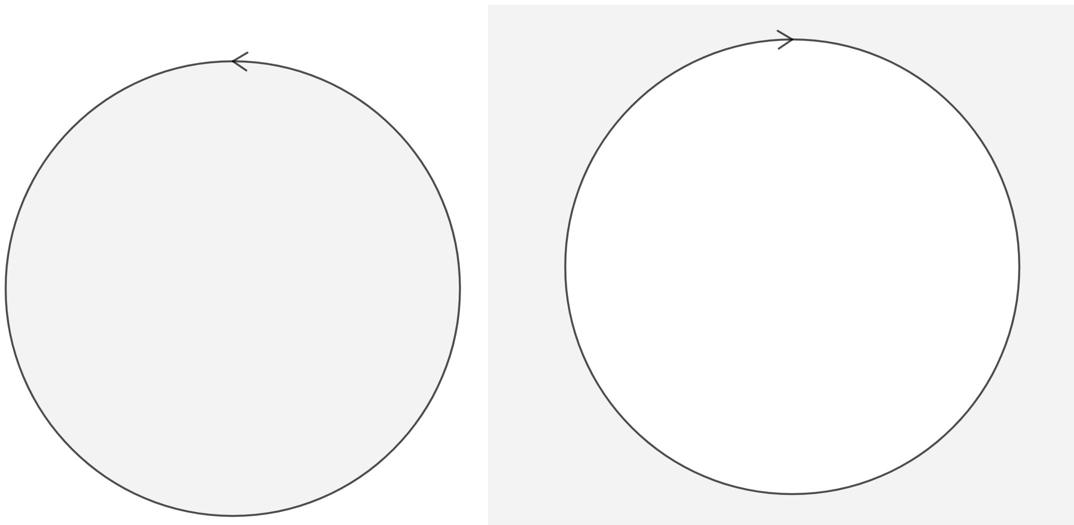

Figure 4.2: An orientation on a circle chooses an interior disk. The interior disk is always the disk to the left as one travels in the direction of the orientation.

Let oriented circles C_1, C_2 belong to $\mathbb{S}^2$, each bounding their respective interior disks D_1 , and D_2. Let O be an oriented circle that is mutually orthogonal to both C_1 and C_2. Label the points of intersection of C_1 and C_2 with D as follows: For C_1, label the points of intersection with D as z_1 and z_2, in order so that the oriented sub-arc z_1z_2 (from z_1 to z_2) is contained in the interior disk $\overline{C}_1$. Denote the points of intersection of C_2 with O as w_1 and w_2 in the same respect.

Definition 4.1.16. *With C_1 and C_2 described as above, the **inversive distance between** C_1 **and** C_2, denoted (C_1, C_2), is defined to be*

$$(C_1, C_2) = 2[z_1, z_2, w_1, w_2] - 1$$

Where $[z_1, z_2, w_1, w_2] = \dfrac{(z_1 - w_1)(z_2 - w_2)}{(z_1 - z_2)(w_1 - w_2)}$ *is the cross ratio.*

The invervise distance is a signed real number since the cross ratio is a signed real number. The ***absolute inversive distance*** $[C_1, C_2] = |(C_1, C_2)|$ is the absolute value of the inversive distance between two circles. From this formula, it is clear that there is a Möbius transformation taking one unoriented pair of circles C_1, C_2 to another pair of unoriented circles C_1', C_2' if and only if $[C_1, C_2] = [C_1', C_2']$.

Although called a “distance,” it is immediately obvious this is not a true distance function. The inversive distance is not positive-definite, and it does not satisfy the triangle inequality. Cross

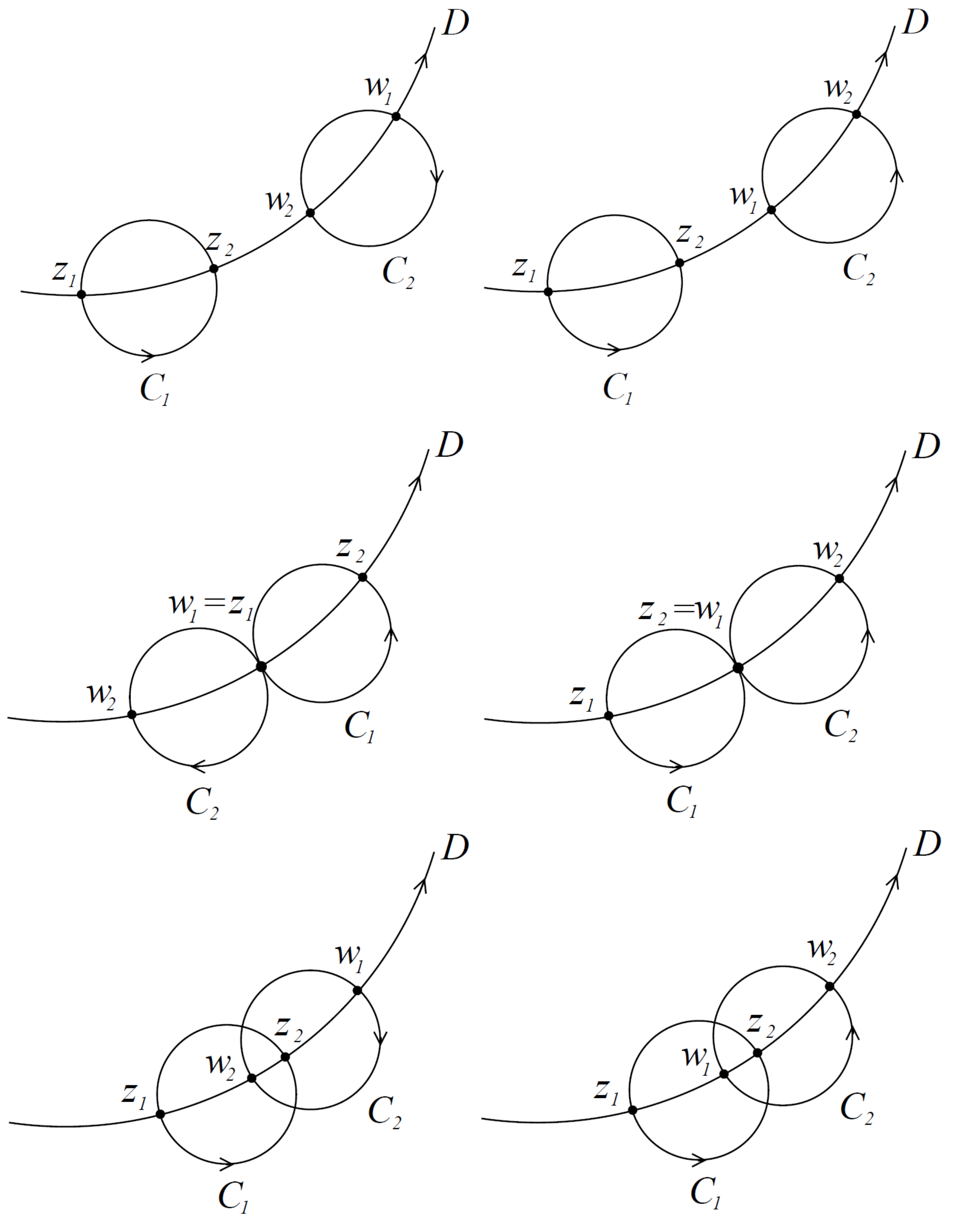

Figure 4.3: From [7]. Six cases of inversive distance using oriented circles. (Top) inversive distance is greater (right) or less than (left) 1; (middle) inversive distance is 1 (right) or −1 (left); (bottom) inversive distance is between 0 and 1 (right) or between −1 and 0 (left).

ratios are preserved by inversive transformations, and thus so are inversive distances. This is used to see that it does not matter which oriented circle D mutually orthogonal to C_1 and C_2 is chosen to define inversive distance. If T is any Möbius transformation that set-wise fixes circles C_1 and C_2, since T takes circles to circles and preserves angles, $T(D)$ will also be a circle orthogonal to both C_1 and C_2. As such, we can go from any circle D orthogonal to C_1 and C_2 to any other circle D' orthogonal via Möbius transformation since the placement of any three points determine such a transformation. Furthermore, orientation of D is irrelevant: the relative orientation of circles C_1 and C_2 with D is what determines the sign of the inversive distance. If the orientation of both C_1 and C_2 are reversed, then $[z_2, z_1, w_2, w_1] = \frac{(z_2 - w_2)(z_1 - w_1)}{(z_2 - z_1)(w_2 - w_1)} = \frac{(z_1 - w_1)(z_2 - w_2)}{-(z_1 - z_2)(-(w_1 - w_2))} = \frac{(z_1 - w_1)(z_2 - w_2)}{(z_1 - z_2)(w_1 - w_2)} = [z_1, z_2, w_1, w_2]$ and so inversive distance is preserved. If only one of C_1 or C_2 has its orientation reversed, however, this changes the sign of (C_1, C_2).

The inversive distance has many equivalent formulations, which are advantageous depending on which information one knows about a given collection of circles. The formula above demonstrates that inversive distance is a conformal invariant. Other formulas are advantageous because they utilize other information commonly used with circles, such as angle, centers, and radii. Within this chapter, we will study three other formulas. We immediately detail two of those formulas below.

Euclidean formula for inversive distance. This formula utilizes the radius and center measurements of given circles, taking advantage of Euclidean distance as an invariants of rigid transformations. Let $C_1 = C(a_1, r_1)$ and $C_2 = C(a_2, r_2)$ be circles in $\hat{\mathbb{R}}^2$ with center a_1, a_2 and radius r_1, r_2 respectively. Then

$$(C_1, C_2) = \frac{|a_1 - a_2|^2 - r_1^2 - r_2^2}{2r_1r_2} \tag{4.6}$$

While this formula is certainly useful in the right circumstances, we don't keep track of centers and radii in our current setting. As such, this formula won't be used frequently in this chapter.

Hyperbolic formula for inversive distance. Since we are primarily interested in the conformal geometry of circles, the well-established hyperbolic formula for inversive distance is of particular interest. For oriented circles C_1 and C_2 in $\hat{\mathbb{C}}$:

$$(C_1, C_2) = \cos\theta$$

Value θ has two different meanings based on how the two circles interact. *Case 1: C_1 and C_2 intersect.* In this case, θ is the Euclidean angle at the point of intersection formed by a unit vector tangent to C_1 at the point of intersection, in the direction of the orientation of C_1, and a unit vector tangent to C_2 at the point of intersection, in the opposite direction of the orientation of C_2. When C_1 and C_2 intersect, the angle of intersection θ has a range of $0 \leq \theta \leq \pi$, which results in an inversive distance (C_1, C_2) with a range of $-1 \leq (C_1, C_2) \leq 1$.

Case 2: C_1 and C_2 do not intersect. Since circles C_1 and C_2 are in $\hat{\mathbb{C}}$, we consider the extended complex plane as the bounary at infinity for the upper half space model of hyperbolic space, $\mathbb{H}^3$, so that unoriented circles C_1 and C_2 can be taken as the boundary of two planes P_1 and P_2 respectively in $\mathbb{H}^3$. Interior disks $\overline{C_1}$ and $\overline{C_2}$ pick an oriented half-space H_1 and H_2 in $\mathbb{H}^3$, where $\partial H_1 = P_1$ an $\partial H_2 = P_2$. Then, $\theta(C_1, C_2) = id_{\mathbb{H}^3}(P_1, P_2)$, so that

$$[C_1, C_2] = \cos id_{\mathbb{H}^3}(P_1, P_2) = \cosh d_{\mathbb{H}^3}(P_1, P_2). \tag{4.7}$$

With consideration of the interior disks, $(C_1, C_2) = \cosh d_{\mathbb{H}^3}(P_1, P_2)$ if halfspaces H_1 and H_2 are disjoint, and $1 < (C_1, C_2) < \infty$ in this case. The inversive distance $(C_1, C_2) = -\cosh d_{\mathbb{H}^3}(P_1, P_2)$ if the halfspaces intersect, in which case $-\infty < (C_1, C_2) < -1$.

Inversive distance has been set up within the context of circles in the Riemann sphere, but inversive distance is a conformal invariant that any pair of n-dimensional spheres carries. The cross ratio, Euclidean, and hyperbolic inversive distance formulas all generalize to higher dimensional spheres in a natural way. We finish this primer on inversive distance with a property from [19].

Theorem 4.1.17 ([19])**.** *For any inversive transformation ϕ and any unoriented spheres S and S',*

$$(\phi(S), \phi(S')) = (S, S'). \tag{4.8}$$

Inversive Ratio. Here, the inversive ratio between a point and two circles (or in higher dimensions, a point and two spheres) is given below in terms of a sequence of inversive distances, along with a method for calculating this invariant using center and radius information. In the next section, this will be related back to the inversive ratio.

Definition 4.1.18. *Let p be a point in $\mathbb{S}^2$, and let C_t be a sequence of oriented circles in $\mathbb{S}^2$ such that C_t is converging to a as $t \to \infty$. Let C and C' be two fixed, oriented circles such that p does not lie on the boundary circles of C or C'. Then*

$$(p, C, C') = \lim_{t \to \infty} \frac{(C_t, C)}{(C_t, C')} \tag{4.9}$$

is called the ***inversive ratio of*** $p, C,$ ***and*** C'.

If $C = C(a, r)$, $C = C(a', r')$, and $C_t = C_t(a_t, r_t)$, then

$$(p, C, C') = \lim_{t\to\infty} \frac{(C_t, C)}{(C_t, C')} = \lim_{t\to\infty} \frac{|a_t - a|^2 - r_t^2 - r^2}{2r_t r} \frac{2r_t r'}{|a_t - a'|^2 - r_t^2 - (r')^2}. \tag{4.10}$$

As $t \to \infty$, observe that in order for C_t to converge to p, we must get that $a_t \to p$ and $r_t \to 0$, so in fact (p, C, C') limits to:

$$(p, C, C') = \frac{(|p - a|^2 - r^2)r'}{(|p - a'|^2 - (r')^2)r}. \tag{4.11}$$

Note that (p, C, C') is always defined, as long as p is not a point on either fixed circle C or C'.

4.1.4 Independence of Circles and Points in $\mathbb{S}^2$

We outline a notion of dependent collections of circles with the goal in mind of developing a notion of independent collections of circles. We begin with the classical notion of a coaxial family of circles, also referred to in literature as a pencil of circles.

Coaxial Families of Circles. A brief summary of a coaxial family of circles is given below. We develop the notion of coaxial families of circles in $\mathbb{E}^2$.

Just as two Euclidean points define a line, two circles define a ***coaxial family of circles***. Two circles, however, may intersect at one point, two points, or not at all. This leads to three different types of coaxial families.

Two points a and b in $\mathbb{E}^2$ generate two mutually orthogonal families of circles. The ***hyperbolic coaxial family***, denoted $H_{ab} = \{h_\lambda : \lambda \in (0, \infty)\}$, is a set of circles separating a from b whose centers lie on a common line called the ***line of centers for*** H_{ab}. The line of centers for H_{ab} is precisely the line through the points a and b. The ***elliptic coaxial family***, denoted $E_{ab} = \{e_\theta : \theta \in [0, \pi)\}$, is the collection of circles passing through a and b; the centers all lie on the ***line of centers for*** E_{ab}. Together, the two mutually orthogonal families are called a ***hyperbolic-elliptic Apollonian system***. The two families must satisfy the following axioms as an Apollonian system:

1. Every circle h_λ of H_{ab} meets every circle e_θ of E_{ab} orthogonally.
2. The circles of H_{ab} are mutually disjoint and partition the punctured space $\mathbb{E}^2 - \{a, b\}$.
3. Each circle h_λ separates a from b.
4. The family E_{ab} consists of all the circles in $\mathbb{E}^2$ that pass through a and b.

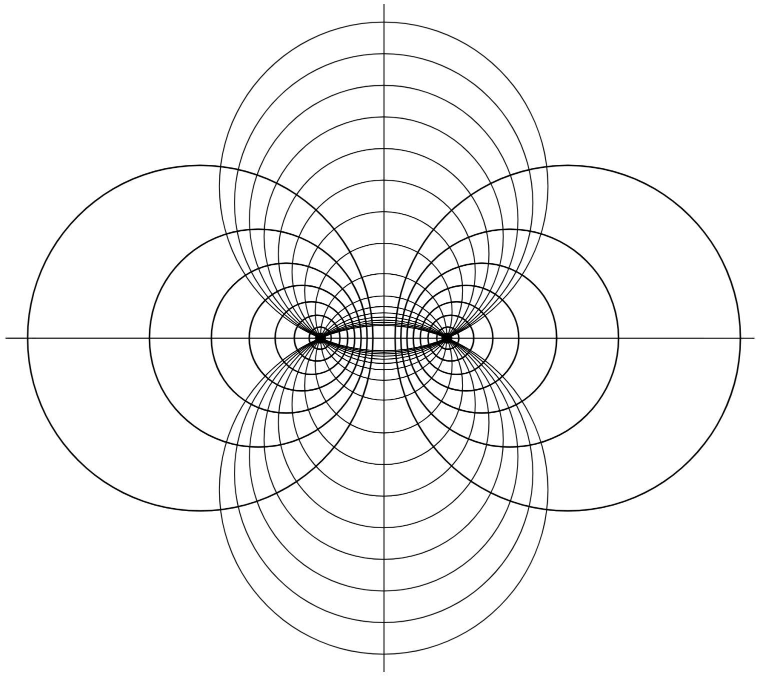

Figure 4.4: The elliptic family of circles intersecting at two points is orthogonal to the hyperbolic family separating the two points.

5. The Euclidean centers of the circles in H_{ab} lie on the circle e_0, the extended line through a and b.

6. The Euclidean centers of the circleses in E_{ab} lie on the circle h_1, the extended perpendicular bisector of the segment $[a, b]$.

As $a \to b$, the hyperbolic-elliptic Apollonian system approaches the ***parabolic Apollonian system***, where H_{ab} and E_{ab} correspondingly limit to ***parabolic coaxial families*** P_l and $P_{l^\perp}$. Here, l is the line determined by a and b, and the line of centers for parabolic coaxial family P_l. The line $l^\perp$ is the line orthogonal to l at point b. P_l is the family of circles which only intersect at point b and are tangent to l; every circle in P_l is orthogonal to every circle in $P_{l^\perp}$.

In order to keep terminology brief, and in keeping with the theme of incidence geometry of circles in this section, we label coaxial families of circles as ***circle-lines***, or ***c-lines*** for short. We add the qualifier parabolic, hyperbolic, or elliptic when referencing a specific type of coaxial family.

Möbius Flows. Coaxial families are the flow lines for ***Möbius Flows***, one-parameter subgroups of $PSL(2,\mathbb{C})$. Let F be a coaxial family; define $\mu_F : \mathbb{R} \to PSL(2,\mathbb{C})$ where $t \mapsto [\mu_F(t)]$, a class of Möbius transformations parametrized by t.

Let F be a hyperbolic coaxial family with fixed points a and b. Then μ_F is an ***elliptic Möbius flow***, a subgroup of Möbius transformations in $PSL(2,\mathbb{C})$ which are conjugate to the standard rotation flow $t \mapsto R_{\lambda t}$, where $R_{\lambda t}$ is the rotation map $z \mapsto e^{\lambda i t} z$, $\lambda \in \mathbb{R}$. By taking a Möbius transformation that sends a to 0 and b to ∞, the direction of the flow assigned to the circles of F is observed. Note that the flow fixes each of the circles of F and preserves the orthogonal elliptic coaxial family $F^\perp$ by taking any circle in $F^\perp$ to another circle in $F^\perp$.

If one takes F to be an elliptic coaxial family with fixed points a and b, then μ_F is a ***hyperbolic Möbius flow***, a subgroup of Möbius transformations in $PSL(2,\mathbb{C})$ which are conjugate to the standard scaling flow $t \mapsto S_{\lambda t}$, in which $S_{\lambda t}$ is the scaling map $z \mapsto e^{\lambda t} z$. Each of the circles of elliptic coaxial family F gets fixed by μ_F and orthogonal hyperbolic coaxial family $F^\perp$ is preserved.

When F is a parabolic coaxial family fixing one point a, μ_F is called a ***parabolic Möbius flow***, a subgroup of Möbius transformations in $PSL(2,\mathbb{C})$ conjugate to the standard translation flow $t \mapsto T_{\lambda t}$, where $T_{\lambda t}$ is the translation map $z \mapsto z + \lambda t$. Each circle in F is fixed, while the orthogonal parabolic coaxial family $F^\perp$ is preserved.

Möbius flows are unique up to linear reparametrization, meaning $[\mu_F(t)] = [\mu_F \circ \lambda] = [\mu_F(ct)]$, where λ is a reparametrization map $t \mapsto ct$, $c \in \mathbb{R} - \{0\}$. The linear reparametrization is called the ***speed*** of a Möbius flow. Note that distinct circles C_1 and C_2 determine a unique coaxial family $\mathcal{A}_{C_1,C_2}$. Any other two distinct circles in $\mathcal{A}_{C_1,C_2}$ will determine the same equivalence class of Möbius flows as C_1 and C_2.

Möbius flows are useful because, given any two circles, a coaxial family F can be determined, and thus, so can a Möbius flow μ_F. Any circle C can be flowed along the flow lines of F to find $\mu_F(C)$ at some time t.

Circle-planes. A ***circle-plane***, or ***c-plane*** for short, in $\mathbb{S}^2$ is a collection of circles in $\mathbb{S}^2$ that do not exclusively belong to a c-line. There are three types of c-planes, and each can be described based upon how any given circle in the c-plane interacts with a ***generating circle*** for the c-plane.

Let C be a circle in $\mathbb{S}^2$. A ***hyperbolic c-plane*** $\mathcal{H}_C$ is the collection of all circles in $\mathbb{S}^2$ which are orthogonal to C, the generating circle of $\mathcal{H}_C$. Such a collection contains hyperbolic, elliptic, and parabolic c-lines.

Let p be a point in $\mathbb{S}^2$. Point p can be thought of as a circle of radius 0. Then a ***parabolic c-plane*** $\mathcal{P}_p$ is the collection of all circles passing through point p. Since every circle in the collection must meet every other circle in the collection at point p, parabolic c-planes exclude hyperbolic c-lines, but include both parabolic and elliptic c-lines.

The last kind of c-plane is called an ***elliptic c-plane***. We begin by developing the ***model elliptic c-plane***. Start with circle $C = \mathbb{S}^1$, the equator of $\mathbb{S}^2$. Define $\mathcal{E}_{\mathbb{S}^1}$ to be the collection of circles intersecting $\mathbb{S}^1$ at its antipodal points. That is, $\mathcal{E}_{\mathbb{S}^1}$ is the collection of great circles in $\mathbb{S}^2$. This collection is the model elliptic c-plane. Now, for any C in $\mathbb{S}^2$, apply a rotation of the sphere so that C is a latitudinal line on $\mathbb{S}^2$. Then use a hyperbolic Möbius flow fixing the north and south pole to flow $\mathbb{S}^1$ to C. Then an ***elliptic c-plane*** $\mathcal{E}_C$ is the collection $\mathcal{E}_{\mathbb{S}^1}$ under that composition of maps.

Independent Collections of Circles in $\mathbb{S}^2$. In Euclidean space, a line can be uniquely determined by two distinct points; a plane is determined by three linearly independent points. Looking at the incidence geometry of circles, a circle-line is determined by two distinct circles. A circle-plane is determined by three *independent* circles in $\mathbb{S}^2$, that is, three circles that do not lie in a common c-line. To elaborate, let C_1, C_2, C_3 be three independent circles in $\mathbb{S}^2$. Three cases arise. In the first case, the circles C_1, C_2, C_3 have a circle O that is commonly orthogonal to all three which happens as long as there is at least one circle that does not overlap the other two by an angle of more than $\pi/3$; in this first case, O is called an ***ortho-circle***, and O is the generating circle for the hyperbolic c-plane $\mathcal{H}_C$. In the second case, all three circles have a common point of intersection p, in which case, the point p generates the parabolic c-plane $\mathcal{P}_p$. In the last case, neither a common orthocircle nor point can be found between the three circles; this case is characterized by each of C_1, C_2, and C_3 overlapping one another, where C_3 separates one intersection point of C_1 and C_2 from the other. In this last case, C_1, C_2, C_3 generate an elliptic c-plane.

Definition 4.1.19. *A collection of four distinct circles* $\{C_1, C_2, C_3, C_4\}$ *in* $\mathbb{S}^2$ *is* ***independent*** *if* C_1, C_2, C_3 *and* C_4 *don't all belong to the same circle-plane.*

A key difference between the behavior of independence as a property of circles and independence as a property of points in Euclidean space is that independence is a metric invariant of points, while it is not an invariant of a general collection of circles. Any collection of points in Euclidean space isometric to an independent set is itself independent. Inversive distance does not preserve the property of independence for circles. As a quick example, take an independent collection of three circles all with inversive distance 1 to one another, all mutually tangent at three distinct points. Take another collection of three circles, this time, lying in a parabolic c-line. The collections are iso-inversive to one another, but the latter collection is not independent.

Despite this, independence is incredibly useful in studying the rigidity of circle configurations. We immediately begin using independence of circles as a tool for determining when a collection of circles is Inversive-equivalent and Möbius-equivalent.

Lemma 4.1.20. *Let C_1, C_2, C_3 be three distinct, independent, unoriented circles, respectively, in $\mathbb{S}^2$, and let f and g be two inversive transformations such that $f(C_i) = g(C_i)$ for $i = 1, 2, 3$. Then, either $f = g$, or else $f = I_C \circ g$, where I_C is an inversion through a circle.*

Proof. Let C_1, C_2, C_3 be three independent unoriented circles. Let f, g be two Inversive transformations, and assume $f(C_i) = g(C_i)$ for $i = 1, 2, 3$. We consider three cases.

Case 1*: $\{C_1, C_2, C_3\}$ determine an elliptic c-plane.* Then each circle intersects the other two circles in two points, respectively, for a total of six intersection points. These points do not lie on a common circle. Therefore, $g^{-1} \circ f$ is an inversive transformation fixing six points not lying on a common circle, so $g^{-1} \circ f$ must be the identity, meaning $f = g$.

Case 2*: $\{C_1, C_2, C_3\}$ determine a parabolic c-plane.* Then there is one common point of intersection p between C_1, C_2, C_3. Since the three circles are independent, they do not belong to a common c-line, and so at least one circle must intersect the other two circles in one other point besides p, respectively, for a total of at least three intersection points. Without loss of generality, let C_1 be a circle intersecting C_2 in p and one other point q, and intersecting C_3 in p and one other point q', where necessarily, $q \neq q'$. Then $g^{-1} \circ f$ fixes points p, q, q'. The only inversive transformations fixing three points is the identity, and the inversion through the circle determined by three points. So either $g^{-1} \circ f$ is the identity map, or $g^{-1} \circ f$ is the inversion through the circle C_1, meaning either $f = g$ or $f = I_{C_1} \circ g$.

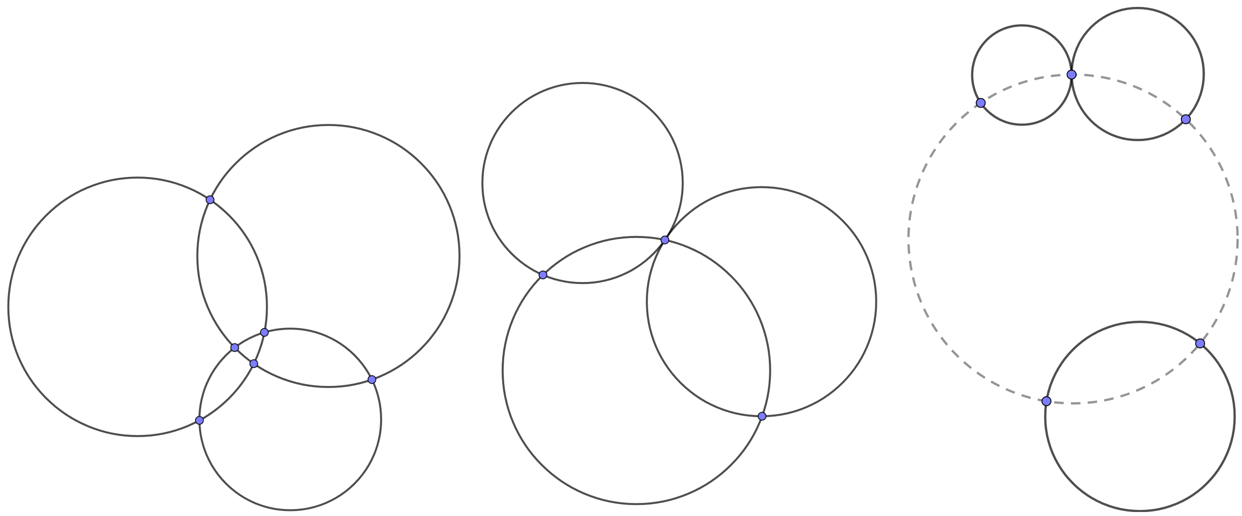

Figure 4.5: An example of Case 1 (left), Case 2 (center), and Case 3 (right). In Case 3, the ortho-circle is dashed.

Case 3: $\{C_1, C_2, C_3\}$ determine a hyperbolic c-plane. Then C_1, C_2, C_3 are mutually orthogonal to a circle O, for a total of at least three intersection points between O and the collection of circles $\{C_1, C_2, C_3\}$. inversive transformation $g^{-1} \circ f$ fixes C_1, C_2, C_3, O, and the six intersection points. Therefore, either $g^{-1} \circ f$ is the identity map, or $g^{-1} \circ f = I_O$, where I_O is the inversion through circle O. □

Note in Case 2 that the only time $g^{-1} \circ f = I_{C_1}$ is when C_1 is mutually orthogonal to C_2 and C_3; otherwise, an inversion through C_1 would not fix C_2 and C_3.

Corollary 4.1.21. *Let C_1, C_2, C_3 be three distinct, independent circles in $\mathbb{S}^2$, and let f and g be two Möbius transformations such that $f(C_i) = g(C_i)$ for $i = 1, 2, 3$. Then $f = g$.*

4.2 Correspondence Between Objects in $\mathbb{S}^2$ and Objects in Lorentz Space $\mathbb{R}^4$

In this section, we carefully lay out the correspondence between the geometry of points and spheres in $\mathbb{S}^{n-1}$, and the geometry of subspaces in $(n+1)$-dimensional Lorentz Space. This dictionary is established through n-dimensional hyperbolic space sitting in Lorentz Space, $\mathbb{R}^{n+1}$. We have already established the presence of the Hyperboloid model $\mathcal{H}^n$ in $\mathbb{R}^{n+1}$; we now focus primarily on the ***Klein Model of hyperbolic n-space***, which we refer to as $\mathbb{H}^n$. First, identify the

$(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ with the collection of x in $\mathbb{R}^{n+1}$ such that $x = (x_1, \ldots, x_n, 1)$, where $x_1^2 + \ldots, x_n^2 = 1$. This is exactly the intersection of the light cone C^n with $x_{n+1} = 1$. Consider the map ϕ

$$(x_1, \ldots, x_n) \mapsto \left(\frac{2x_1}{1-|x|^2}, \ldots, \frac{2x_n}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2} \right), \tag{4.12}$$

from $\mathcal{H}^n$ to the unit ball $\mathbb{B}_1^n = \mathbb{H}^n$, with boundary $\mathbb{S}^{n-1}$, containing point $(0, \ldots, 0, 1)$. This map is a Lorentz transformation, and is a hyperbolic isometry. Furthermore, for positive time-like unit vectors v_x through a point x in $\mathcal{H}^n$, note that map ϕ takes vector v_x and rescales it to a positive time-like vector $\frac{1}{\lambda_{n+1}} v_x$ which has $x_{n+1} = 1$ for the last coordinate. Note, additionally, that any $(m+1)$-dimensional time-like subspaces of $\mathbb{R}^{n+1}$ intersecting $\mathcal{H}^n$ as m-planes also intersect $\mathbb{H}^n$ as m-planes. This convenient placement of $\mathbb{S}^{n-1}$ as the ideal boundary of $\mathbb{H}^n$ within $\mathbb{R}^{n+1}$ provides a setting in which to draw a parallel between collections of spheres and points in $\mathbb{S}^{n-1}$ and subspaces within $\mathbb{R}^{n+1}$.

It's already been noted that Inv($\mathbb{S}^{n-1}$) is isomorphic to $O^+(n,1)$.

4.2.1 Spheres in $\mathbb{S}^{n-1}$ Correspond to Space-Like Unit Vectors

Let S be an unoriented sphere in $\mathbb{S}^{n-1}$. Then S is the intersection of an n-dimensional time-like subspace V in $\mathbb{R}^{n+1}$. There is a 1-dimensional subspace V^L of space-like vectors Lorentz orthogonal to V. When S is unoriented, the positive space-like unit vector v in V^L is assigned as default space-like vector corresponding to S.

When S is oriented, with an interior ball B, the space-like unit vector representing S with the correct orientation is chosen in the following manner. First, assume S is not a great sphere in $\mathbb{S}^{n-1}$. Consider the intersection of line V^L at $x_{n+1} = 1$. This intersection is a point c outside $\mathbb{H}^n \cup \mathbb{S}^{n-1}$. Point c is the vertex for an $(n-1)$-dimensional cone C_S which intersects $\mathbb{S}^{n-1}$ as sphere S: every line in C_S is tangent to $\mathbb{S}$ at a unique point in S. Subspace V separates $\mathbb{R}^{n+1}$ in two half-spaces. If B is the ball corresponding to the half-space of V containing point c, use the positive space-like unit vector v_+ in V^L to represent S. Otherwise, represent S with the negative space-like unit vector v_- in V^L.

For simplicity, when S is the sphere in $\mathbb{S}^{n-1}$ that is the intersection of n-dimensional time-like V, and w is a vector in V^L, we say that S is ***Lorentz orthogonal*** to w.

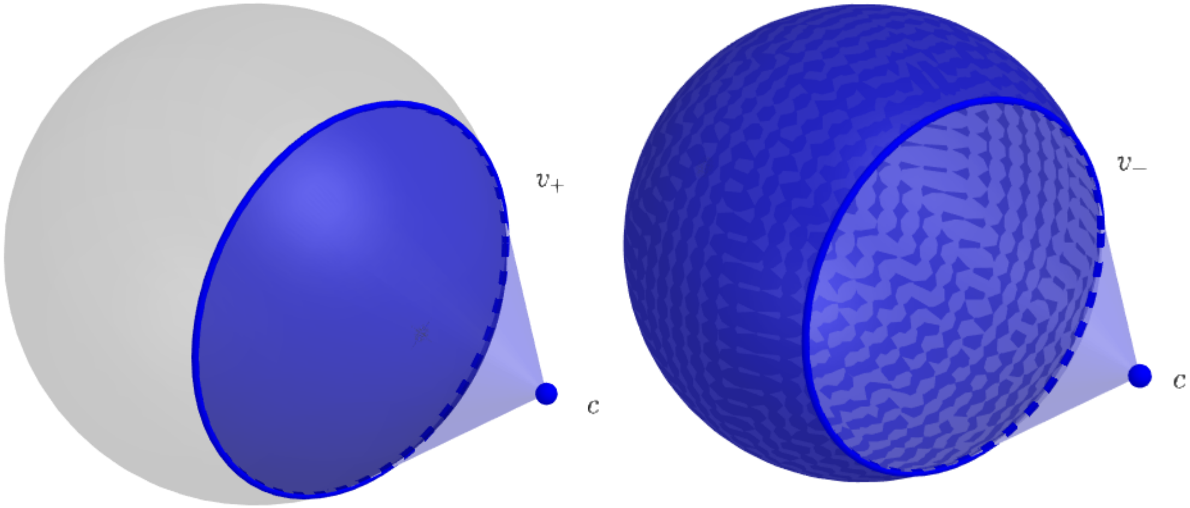

Figure 4.6: In this picture, a sphere S refers to a circle in $\mathbb{S}^2$, and a ball refers to a disk. Let V_S be the time-like subspace supporting S. For an oriented sphere, interior ball B is the intersection of a halfspace of V_S with $\mathbb{S}^{n-1}$. If the halfspace contains point c, represent the oriented sphere with the positive space-like unit vector (left); else, the oriented sphere is represented with the negative space-like unit vector (right).

Inversive Distance Correspondence. We first consider the inversive distance for intersecting spheres. Let P and Q be hyperplanes in $\mathbb{H}^n$, either intersecting in $\mathbb{H}^n$, or whose boundaries meet only once at infinity. While the Klein Model $\mathbb{H}^n$ is not a conformal model of hyperbolic space, the angle of intersection between hyperplanes in $\mathbb{H}^n$ is given by the angle of intersection θ between the spheres S_P and S_Q that are the boundary of each respective hyperplane in $\mathbb{S}^{n-1}$. In Chapter 3, we saw that $\cos\theta$ is the Lorentz inner product between the space-like unit vectors Lorentz orthogonal to P and Q respectively. Compare this with using the hyperbolic inversive distance formula for spheres S_P and S_Q in $\mathbb{S}_1^{n-1}$, equipped with interior balls $\overline{S_P}$ and $\overline{S_Q}$ respectively. Then

$$(S_P, S_Q) = -\cos\theta = -\langle v_P, v_Q\rangle, \tag{4.13}$$

where $0 \leq \theta \leq \pi$ is the angle of intersection between S_P and S_Q given by the space-like angle between v_P and v_Q.

When P and Q are hyperplanes in $\mathbb{H}^n$ which are disjoint in $\mathbb{H}^n$, and whose boundaries are disjoint in $\mathbb{S}^{n-1}$, the hyperbolic distance $d_{\mathbb{H}}(P, Q)$ is used both for the inversive distance between

S_P and S_Q, with the caveat that inversive distance is positive only when v_P and v_Q are oppositely oriented tangent vectors for N, the unique hyperbolic line between P and Q, and positive all other times. This gives us that

$$[S_P, S_Q] = \cosh d_{\mathbb{H}}(P, Q) = |\langle v_P, v_Q \rangle|. \tag{4.14}$$

In general, our *Lorentz inversive distance formula* is

$$(S_P, S_Q) = -\langle v_P, v_Q \rangle \tag{4.15}$$

making it the simplest formula for inversive distance thus far.

4.2.2 Points Correspond to Light-Like Lines

This correspondence has already been utilized in the previous chapter. Specifically, it was already stated that light-like lines are used to represent points $(x_1, \ldots, x_n, 1)$ in the light cone C^n, which are identified as points in the ideal boundary of $\mathcal{H}^n$. These are the points such that $x_1^2 + \ldots + x_n^2 - 1 = 0$, which are exactly the points in $\mathbb{S}_1^{n-1}$. Hence, any point in $\mathbb{S}^{n-1}$ can be represented by a light-like line and vice versa.

Absolute Cross Ratio Correspondence. We have already introduced the concept of an absolute cross-ratio of light-like lines, taken from [13]. Here, we examine the relationship between the two cross-ratios. This observation is stated by [13], but here a different explanation is provided.

Lemma 4.2.1. *([13]) If a point a_i in $\mathbb{S}^{n-1}$ corresponds to the line ℓ_i under ψ, then*

$$|\ell_1, \ell_2, \ell_3, \ell_4| = |a_1, a_2, a_3, a_4|^2. \tag{4.16}$$

Proof. Let a_1, a_2, a_3, a_4 be points in $\mathbb{S}^{n-1}$, and let $\ell_1, \ell_2, \ell_3, \ell_4$ be the respective lines corresponding to the points under ψ. Choose light-like vectors $v_i = (a_{i1}, \ldots, a_{in}, 1)$ for each ℓ_i. Then

$$
\begin{aligned}
|\ell_1, \ell_2, \ell_3, \ell_4| &= \frac{\langle v_1, v_3\rangle\langle v_2, v_4\rangle}{\langle v_1, v_2\rangle\langle v_3, v_4\rangle} \\
&= \frac{(a_{11}a_{31} + \ldots + a_{1n}a_{3n} - 1)(a_{21}a_{41} + \ldots + a_{2n}a_{4n} - 1)}{(a_{11}a_{21} + \ldots + a_{1n}a_{2n} - 1)(a_{31}a_{41} + \ldots + a_{3n}a_{4n} - 1)} \\
&= \frac{-2(a_{11}a_{31} + \ldots + a_{1n}a_{3n} - 1)(-2)(a_{21}a_{41} + \ldots + a_{2n}a_{4n} - 1)}{-2(a_{11}a_{21} + \ldots + a_{1n}a_{2n} - 1)(-2)(a_{31}a_{41} + \ldots + a_{3n}a_{4n} - 1)} \\
&= \frac{(a_{11}^2 - 2a_{11}a_{31} + a_{31}^2 + \ldots + a_{1n}^2 - 2a_{1n}a_{3n} + a_{3n}^2)(a_{21}^2 - 2a_{21}a_{41} + a_{41}^2 + \ldots + a_{2n}^2 - 2a_{2n}a_{4n} + a_{4n}^2)}{(a_{11}^2 - 2a_{11}a_{21} + a_{21}^2 + \ldots + a_{1n}^2 - 2a_{1n}a_{2n} + a_{2n}^2)(a_{31}^2 - 2a_{31}a_{41} + a_{41}^2 + \ldots + a_{3n}^2 - 2a_{3n}a_{4n} + a_{4n}^2)} \\
&= \frac{[(a_{11} - a_{31})^2 + \ldots + (a_{1n} - a_{3n})^2][(a_{21} - a_{41})^2 + \ldots + (a_{2n} - a_{4n})^2]}{[(a_{11} - a_{21})^2 + \ldots + (a_{1n} - a_{2n})^2)][(a_{31} - a_{41})^2 + \ldots + (a_{3n} - a_{4n})^2]} \\
&= \frac{|a_1 - a_3|^2|a_2 - a_4|^2}{|a_1 - a_2|^2|a_3 - a_4|^2} = |a_1, a_2, a_3, a_4|^2.
\end{aligned}
$$

□

4.2.3 The Inversive Ratio is the Lorentz Ratio

With the equality between inversive distance of spheres and Lorentz inner product of space-like unit vectors established, the following statement should come as no surprise.

Lemma 4.2.2. *Let a be a point in $\mathbb{S}^{n-1}$, S_t a sequence of spheres converging toward point a as $t \to \infty$, and S_u, S_w two fixed spheres in $\mathbb{S}^{n-1}$, such that a is not in the boundary spheres S_u or S_w. Let ℓ_a be the light-like line in $\mathbb{R}^{n+1}$ through point a, let v_t be the sequence of space-like unit vectors converging to ℓ, Lorentz orthogonal to S_t and let u and w be the space-like unit vectors Lorentz orthogonal to S_u and S_v. Then*

$$
(a, S_u, S_w) = (\ell_a, u, w). \tag{4.17}
$$

One interesting aspect to note is that the Euclidean inversive distance formula gives an alternate formula for finding the inversive ratio in this instance, should radius and center data be available for use.

4.2.4 Extrinsic View of Circle-Planes in $\mathbb{S}^2$

At this point, since the correspondences between spheres in $\mathbb{S}^{n-1}$ and hyperplanes in $\mathcal{H}^n$ and space-like unit vectors in $\mathbb{R}^{n+1}$ have been established, it is clear that the characterization of collections of n hyperplanes in $\mathcal{H}^n$ corresponding to linearly independent collections of n space-like unit vectors that was laid out in Chapter 3, leads to a characterization of collections of n spheres in $\mathbb{S}^{n-1}$ which correspond to linearly independent collections of n space-like unit vectors. This correspondence does lead to a rigidity result for spheres and points in $\mathbb{S}^{n-1}$. This result is basically a restatement of the main theorem in Chapter 3, so it is left to the reader. Instead, our focus now turns to a statement of the specific case when $n = 3$, where we may use the language of independence of circles. This case is of particular interest because, as we will see, the notion of independence of circles can be used to gain rigidity results for specialty collections of circles, such as inversive distance circle packings. With this use in mind, in this section we focus on outlining the corresondence between 3-dimensional collections of space-like vectors in $\mathbb{R}^4$ and the corresponding collections of circles in $\mathbb{S}^2$.

Hyperbolic c-planes.

Lemma 4.2.3. *A collection of circles $\{C_1, \ldots, C_k\}$, $k \geq 3$ generates a hyperbolic c-plane if and only if the corresponding collection of space-like unit vectors $\{v_1, \ldots, v_k\}$ spans a time-like 3-dimensional subspace.*

This is true because collection $\{C_1, \ldots, C_k\}$ is a collection of circles in $\mathbb{S}^2$, acting as the boundaries at infinity to hyperplanes $P_1, \ldots, P_k$ respectively in $\mathcal{H}^3$. Since the circles lie in a hyperbolic c-plane, there is a unique circle O commonly orthogonal to all C_i, for $i = 1, \ldots, k$, acting as the boundary for hyperplane P_O commonly orthogonal to each P_i, so collection $\{P_1, \ldots, P_k\}$ is a collection of hyperplanes satisfying the conditions of Lemma 3.1.22.

Elliptic c-planes.

Lemma 4.2.4. *A collection of circles $\{C_1, \ldots, C_k\}$, with $k \geq 3$, generates an elliptic c-plane if and only if the corresponding collection of space-like unit vectors $\{v_1, \ldots, v_k\}$ spans a space-like 3-dimensional subspace.*

Using the definition of an elliptic c-plane, there is an inversive transformation taking $\{C_1, \ldots, C_k\}$ to a collection of great circles in $\mathbb{S}^2$, and thus, respective hyperplanes $P_1, \ldots, P_k$ are taken to hyperplanes $\Pi_1, \ldots, \Pi_k$, where $(0, \ldots, 0, 1)$ is the only point common to all Π_i. Thus, collection $\{P_1, \ldots, P_k\}$ fit the conditions of Lemma 3.1.21.

Parabolic c-planes.

Lemma 4.2.5. *A collection of circles $\{C_1, \ldots, C_k\}$, where $k \geq 3$, generates a parabolic c-plane if and only if the corresponding collection of space-like unit vectors $\{v_1, \ldots, v_k\}$ spans a light-like 3-dimensional subspace.*

Since $\{C_1, \ldots, C_k\}$ lie on a parabolic c-plane, hyperplanes $P_1, \ldots, P_k$ all meet at unique point at infinity, and there is not a hyperplane P_O distinct from P_i that is orthogonal to every P_i for $i = 1, \ldots, k$. Thus, the conditions of Lemma 3.1.24 are met.

Lemma 4.2.6. *Every c-plane is exclusively parabolic, hyperbolic, or elliptic.*

This is seen easily by considering the generating circles of each kind of c-plane, where the generating circle of a parabolic c-plane is a point, and elliptic c-planes have imaginary generating circles. Each kind of circle is supported by a three-dimensional subspace of $\mathbb{R}^4$, which is time-like if the c-plane is hyperbolic, light-like if the c-plane is parabolic, and space-like if the c-plane is elliptic.

4.2.5 Independent Collections of Points and Circles

As stated previously, any collection of four points corresponds to a collection of light-like lines which is maximally independent in $\mathbb{R}^4$.

Lemma 4.2.7. *A collection of four circles $\{C_1, C_2, C_3, C_4\}$ in $\mathbb{S}^2$ is independent if and only if space-like unit vectors $\{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^{n+1}$, where v_i is Lorentz orthogonal to C_i for each $i = 1, 2, 3, 4$.*

4.2.6 Consistently Oriented Collections of Circles

Definition 4.2.8. *Let $\{C_i\}$ and $\{C_i'\}$ be two collections of oriented circles in $\mathbb{S}^2$, and let $\{O_j\}$ and $\{O_j'\}$ respectively be the collection of orthocircles accompanying each collection; assume that no circle in any collection is a great circle. Let $\{v_i\}$ and $\{v_i'\}$ respectively be the collection of space-like unit vectors corresponding to $\{C_i\}$ and $\{C_i'\}$; let $\{w_j\}$ and $\{w_j'\}$ be the collection of space-like unit vectors corresponding to $\{O_j\}$ and $\{O_j'\}$ respectively. Collections $\{C_i\}$ and $\{C_i'\}$ are* ***oriented consistently*** *whenever each v_i and each w_j is positive if and only v_i' is positive, and the sign of $\langle v_i, w_j \rangle$ is the same as the sign of $\langle v_i', w_j' \rangle$ for every pair i, j.*

Lemma 4.2.9. *There is a Möbius transformation taking oriented circle pair C_1, C_2 to oriented circle pair C_1', C_2' if and only if $(C_1, C_2) = (C_1', C_2')$ and the circle pairs are oriented consistently.*

Lemma 4.2.10. *Let $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ be two consistently oriented collections of independent circles in $\mathbb{S}^2$, where $(C_i, C_j) = (C'_i, C'_j)$ for each distinct pair i, j. Then there is a Möbius transformation f such that $f(C_i) = C'_i$ for $i = 1, 2, 3$.*

Proof. Let $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ be as above. By 4.2.9, there is a Möbius transformation f where $f(C_1) = C'_1$ and $f(C_2) = C'_2$. Observe that $(f(C_3), C'_i) = (C'_3, C'_i)$ for $i = 1, 2$. This means there is a Möbius transformation g in the Möbius flow fixing C'_1 and C'_2 which takes $f(C_3)$ to C'_3. The composition $g \circ f$ is a Möbius transformation taking one collection of oriented circles to the other. □

Lemma 4.2.11. *Let $\{C_1, C_2, C_3, C_4\}$ and $\{C'_1, C'_2, C'_3, C'_4\}$ be two independent collections of circles, oriented consistently, where $(C_i, C_j) = (C'_i, C'_j)$ for distinct pairs $1 \leq i, j \leq 4$. Then there is a unique Möbius transformation f such that $f(C_i) = C'_i$, for all $i = 1, 2, 3, 4$.*

Proof. Assume without loss of generality that neither collection contains great circles, and no collection has three circles orthogonal to a great circle. Move the collections by respective Möbius transformations if necessary so this is true. Let $\{v_1, v_2, v_3, v_4\}$ and $\{v'_1, v'_2, v'_3, v'_4\}$ be the space-like unit vectors corresponding to each respective collection of independent circles. Label the orthocircles of each collection as $\{O_j\}$ and $\{O'_j\}$, respectively, with corresponding collections of space-like unit vectors $\{w_j\}$ and $\{w'_j\}$. Since the two collections of circles are independent, the space-like vectors form a basis of $\mathbb{R}^4$, and there is a unique Lorentz transformation ϕ such that $\phi(v_i) = v'_i$ for $i = 1, 2, 3, 4$. Since the two independent subcollections of circles are oriented consistently, ϕ takes positive basis vectors to positive basis vectors, so ϕ is an inversive transformation.

By Lemma 4.2.10, there is a Möbius transformation σ such that $\sigma(C_1) = C'_1, \sigma(C_2) = C'_2$, and $\sigma(C_3) = C'_3$, so either $\sigma = \phi$ or $\sigma = I_C \circ \phi$, where I_C is an inversion. By Lemma 4.1.20, the latter only happens when $\{C_1, C_2, C_3\}$ (and by extension $\{C'_1, C'_2, C'_3\}$) lie in a hyperbolic c-plane, in which case, either $\sigma = \phi$ or $\sigma = I_{O'} \circ \phi$, where $I_{O'}$ is the inversion through circle O' orthogonal to C'_1, C'_2, C'_3. Suppose $\sigma = I_{O'} \circ \phi$. Then $\phi(C_4) = C'_4 \neq \sigma(C_4) = I_{O'}(C'_4)$. Let u_4 by the space-like unit vector corresponding to $\sigma(C_4)$, and let w' be the space-like unit vector corresponding to circle O'. Then $\langle u_4, w' \rangle = -\langle v_4, w' \rangle$, contradicting the assumption that the independent subcollections are oriented consistently. Therefore, $\sigma = \phi$. □

4.2.7 Rigidity of Points and Circles in $\mathbb{S}^2$

Theorem 4.2.12. *Let $\{a_\gamma : \gamma \in \mathcal{A}\}$ and $\{a'_\gamma : \gamma \in \mathcal{A}\}$ be two collections of distinct points in $\mathbb{S}^{n-1}$, each with subcollections of $n+1$ points $\{a_i\}$ and $\{a'_i\}$, respectively, such that $|a_i, a_j, a_k, a_l| = |a'_i, a'_j, a'_k, a'_l|$ for every distinct unordered 4-tuple $1 \leq i, j, k, l \leq n+1$. Then,*

$$|a_\gamma, a_i, a_j, a_k| = |a'_\gamma, a'_i, a'_j, a'_k|, \tag{4.18}$$

for every distinct unordered triplet i, j, k in the independent subcollection, and all γ, if and only if there is a unique inversive transformation Φ such that $\Phi(a_\gamma) = a'_\gamma$ for all $\gamma \in \mathcal{A}$.

This theorem is a direct corollary of theorem 2.3.12, where each point a_γ corresponds to a light-like line ℓ_γ, and each absolute cross ratio of points corresponds to a cross ratio of light-like lines. Any collection of $n+1$ points in $\mathbb{S}^{n+1}$ automatically corresponds to a collection of light-like lines whose vectors span $\mathbb{R}^{n+1}$.

Theorem 4.2.13. *Let $\{C_\alpha, p_\beta : \alpha, \beta \in \mathcal{A}\}$ and $\{C'_\alpha, p'_\beta : \alpha, \beta \in \mathcal{A}\}$ be two collections of oriented circles and points, respectively, in $\mathbb{S}^2$. Suppose each collection has an independent subcollection of four circles, $\{C_1, C_2, C_3, C_4\}$ and $\{C'_1, C'_2, C'_3, C'_4\}$, resp., none of which are great circles, where p_β (resp. p'_β) are not points in C_i (resp. C'_i) for each $i = 1, 2, 3, 4$, and where $(C_i, C_j) = (C'_i, C'_j)$ for each distinct pair $1 \leq i, j \leq 4$. Then there is a unique inversive transformation ϕ such that one of the following holds: either $\phi(C_\alpha) = C'_\alpha$ and $\phi(p_\beta) = p'_\beta$ for each α, β in $\mathcal{A}$, or else $\phi(\overline{C_\alpha}) = C'_\alpha$ and $\phi(p_\beta) = p'_\beta$ for each α, β in $\mathcal{A}$, if and only if $(C_\alpha, C_i) = (C'_\alpha, C'_i)$ for each distinct pair α, i in $\mathcal{A}$ and $(p_\beta, C_i, C_j) = (p'_\beta, C'_i, C'_j)$ for each distinct triple β, i, j in $\mathcal{A}$.*

Proof. Let $\{C_\alpha, p_\beta : \alpha, \beta \in \mathcal{A}\}$ and $\{C'_\alpha, p'_\beta : \alpha, \beta \in \mathcal{A}\}$, be two collections of oriented circles, C_α, C'_α resp, and points, p_β, p'_β resp, in $\mathbb{S}^2$, each with respective independent collections of 4 circles, $\{C_1, C_2, C_3, C_4\}$ and $\{C'_1, C'_2, C'_3, C'_4\}$, such that $(C_i, C_j) = (C'_i, C'_j)$ for each distinct pair $1 \leq i, j \leq 4$. Assume that $(C_\alpha, C_i) = (C'_\alpha, C'_i)$ for every distinct pair of α, i, and $(p_\beta, C_i, C_j) = (p'_\beta, C'_i, C'_j)$ for every distinct triple β, i, j. For each C_α, C'_α, let Π_α, Π'_α respectively be the n-dimensional time-like subspaces in $\mathbb{R}^{n+1}$ intersecting $\mathbb{S}^2$ as C_α, C'_α. Let v_α, v'_α be the corresponding space-like unit vectors Lorentz orthogonal to Π_α, Π'_α respectively. Let ℓ_β, ℓ'_β be the light-like lines through each point p_β, p'_β respectively. Then $\langle v_\alpha, v_i\rangle = \langle v'_\alpha, v'_i\rangle$, and $(\ell_\beta, v_i, v_j) = (\ell'_\beta, v'_i, v'_j)$, so by Theorem 2.3.8, there is a unique Lorentz transformation ϕ such that $\phi(v_\alpha) = v'_\alpha$ and $\phi(\ell_\beta) = \ell'_\beta$ for all α, β in $\mathcal{A}$. If ϕ is a positive Lorentz transformation, then ϕ restricts to an inversive transformation. If ϕ is a negative Lorentz transformation, then $-\phi$ restricts to an inversive transformation and $-\phi(-v_\alpha) = v'_\alpha$ for every α in $\mathcal{A}$. $\square$

Adding in the requirement that the independent subcollections must be consistently oriented guarantees that the two collections are Möbius-congruent.

Corollary 4.2.14. *Let $\{C_\alpha, p_\beta : \alpha, \beta \in \mathcal{A}\}$ and $\{C'_\alpha, p'_\beta : \alpha, \beta \in \mathcal{A}\}$ be two collections with assumptions set up as in Theorem 4.2.13. Further suppose that the subcollections of independent circles $\{C_1, C_2, C_3, C_4\}$ and $\{C'_1, C'_2, C'_3, C'_4\}$ are oriented consistently. Then ϕ is a Möbius transformation.*

Corollary 4.2.14 is a direct consequence of Lemma 4.2.11.

For those that study configurations of circles via inversive geometry, it is atypical to have this amount of inversive distance information for a configuration. It is more representative to know inversive distance information in a polyhedral graph or triangulation pattern, ie, where there known inversive distance information is distributed more evenly across the configuration. With this in mind, we now use notion of independence in the setting of inversive distance circle packings.

4.3 Circle-Polyhedra

In Chapter 1, the connection between Euclidean polyhedra and configurations of circles was discussed. In this way, aspects of rigidity theory can be applied in understanding the existence and rigidity of configurations of circles. In this section, the concept of a *circle-polyhedron* is introduced, and the main theorems from [5] are stated.

4.3.1 Preliminary Definitions and Observations

Definition 4.3.1. *Let G be a graph, ie, a set of vertices $V = V(G)$ and simple edges $E = E(G)$. A* ***circle framework with adjacency graph*** *G or* ***c-framework*** *for short, is a collection $\mathcal{C} = \{C_u : u \in V(G)\}$ of oriented circles in $\mathbb{S}^2$ indexed by the vertex set of G. This is denoted $G(\mathcal{C})$.*

When uv is an edge in $E(G)$, we say that oriented circles C_u and C_v are **adjacent**. For the purposes of this dissertation, our focus lies with inversive distance circle packings: circle frameworks with adjacency graphs which are triangulations of either $\mathbb{D}$ or $\mathbb{S}^2$. We will only consider collections with finitely many circles.

Definition 4.3.2. *An* ***edge-label*** *is a real-valued function* $\Gamma : E(G) \to \mathbb{R}$ *defined on the edge set of* G*, and* G *together with an edge-label* Γ *is denoted* G_Γ *and called an* ***edge-labeled graph****. The c-framework* $G(\mathcal{C})$ *is a* ***circle realization*** *of the edge-labeled graph* G_Γ *provided* $(C_u, C_v) = \Gamma(uv)$ *for every edge* uv *of* G*; this is denoted as* $G_\Gamma(\mathcal{C})$*.*

There are many qualifiers that may be attached to a *c*-framework. They are listed below.

Definition 4.3.3. *A c-framework* $G(\mathcal{C})$*...*

(i) is ***edge-uncoupled*** *if each pair of adjacent circles has inversive distance greater than* -1*;*

(ii) is ***edge-segregated*** *if each pair of adjacent circles has an inversive distance greater than or equal to* 0*;*

(iii) is ***edge-separated*** *if each pair of adjacent circles has an inversive distance greater than* 1 *(each pair of adjacent companion disks is disjoint);*

(iv) is ***non-unitary*** *if the inversive distance of each pair of adjacent circles is not equal to* ± 1*;*

(v) has ***deep overlaps*** *if there is a pair of adjacent circles such that the inversive distance is less than* 0*.*

In this chapter, three different types of *c*-planes were described. We now turn our attention specifically to hyperbolic *c*-planes as a means of introducing a notion of convexity to collections of circles. Moving forward, a collections of circles being *c-planar* refers to the collection belonging to a hyperbolic *c*-plane.

Definition 4.3.4. *Let* P *be an abstract oriented spherical polyhedron, and let* $G = P^{(1)}$ *be its polyhedral graph. A* ***circle-polyhedron****, or* ***c-polyhedron****, is an edge-uncoupled c-framework* $G(\mathcal{C})$ *such that for each face* $f = u_1...u_n$ *of* P*, the corresponding c-**face*** $\mathcal{C}_f = \{C_{u_1}, ..., C_{u_n}\}$ *is c-planar (but not c-linear). The unique ortho-circle for* $\mathcal{C}_f$ *is denoted* O_f

With the notion of what it means to be a c-polyhedron established, convexity of such a collection is described and then a Cauchy-type rigidity for c-polyhedra is introduced.

Definition 4.3.5. *Let* $G(\mathcal{C})$ *be a c-polyhedron based on abstract spherical polyhedron* P*. Let* f *be a face of* P*. Then* $G(\mathcal{C})$ *is* ***convex with respect to*** f *if its corresponding ortho-circle* O_f *may be oriented so that every circle in* $\mathcal{C}$ *is segregated from* O_f*.* $G(\mathcal{C})$ *is* ***convex*** *provided that it is convex with respect to every face of* P*. We avoid any unneccessary pathologies by assuming that the circles corresponding to three consecutive vertices in a face are never coaxial.*

When an ortho-circle O_f may be oriented so that $G(\mathcal{C})$ is convex with respect to f, the ortho-circle is denoted ${O_f}^+$.

Lemma 4.3.6 (Bowers, Bowers and Pratt)**.** *Let $G(\mathcal{C})$ be a convex c-polyhedron based on P. Then either:*

(i) For every oriented face $f = u_1 \dots u_n$ of P, the circles $C_{u_1}, \dots, C_{u_n}$ are met in that order as one progresses around O_f^+ in direction of its orientation, starting at C_{u_1}, or

(ii) For every oriented face $f = u_1 \dots u_n$ of P, the circles $C_{u_1}, \dots, C_{u_n}$ are met in that order as one progresses around O_f^+ in the direction opposite of its orientation, starting at C_{u_1}.

If (ii) occurs, one can always apply the antipodal map to $G(\mathcal{C})$ and reverse the orientation of all circles. In this way, it is standard to assume a convex c-polyhedron for an oriented polyhedral graph P has ortho-circles with orientation consistent with P.

The concept of a proper c-polyhedron is set up.

Definition 4.3.7. *Let $H = \{h_1, ..., h_n\}$ be a set of half-planes in $\mathbb{H}^2$ such that region $P = h_1 \cap \cdots \cap h_n$ is non-empty and the boundary line ℓ_i of each h_i supports a non-empty segment, ray, or line on the boundary of P. The lines ℓ_i for $i = 1, \cdots, n$ are oriented consistent with the orientation P inherits from $\mathbb{H}^2$ so that $\partial^+ h_i = \ell_i$. If P is bounded, then it is a compact convex polygon in $\mathbb{H}^2$. Whether or not P is bounded, we call P a **convex hyperideal polygon**.*

Definition 4.3.8. *The convex hyperideal polygon P determined by the cyclically ordered oriented lines $\ell_1, \cdots, \ell_n$ is said to be **proper** provided the following two conditions are met.*

1. *Any hyperideal vertex, say for instance the hyperbolic line segment $s_{i,i+1}$ meeting the two consecutive lines ℓ_i and ℓ_{i+1} orthogonally, does not meet any other of the oriented lines bordering P. This is equivalent to saying each hyperideal vertex lies in the region P.*

2. *The oriented lines $\ell_1, \cdots, \ell_n$ along with any hyperideal vertices form the boundary of a bounded or compact convex polygon P' contained in P.*

Definition 4.3.9. *Let $G(\mathcal{C})$ be a non-unitary convex c-polyhedron based on the oriented abstract spherical polyhedron P with vertex set $V = V(P)$. For any vertex $v \in V$, give the interior of the companion disk D_v to C_v a complete hyperbolic metric of constant curvature -1 making D_v a model of the hyperbolic plane with its circle at infinity. Let $f_1, ...f_n$ be the faces adjacent to v*

ordered cyclically about v with respect to the orientation of P. Let ℓ_i be the hyperbolic line in D_v determined by the orthogonal intersection $D_v \cap {O_{f_i}}^+$, but oriented oppositely to that of the ortho-circle ${O_{f_i}}^+$. Then $G(\mathcal{C})$ is ***proper*** *or* ***compact at*** *v if the oriented lines $\ell_1, ..., \ell_n$ are the support lines of a proper convex hyperideal polygon $P(v)$ in D_v. The c-polyhedron $G(\mathcal{C})$ then is* ***proper*** *or* ***compact*** *provided it is proper at each of its vertices.*

This definition of a proper *c*-polyhedron is analogous to that of a bounded Euclidean polyhedron. Cauchy's original rigidity theorem uses bounded polyhedra in the setup. Just like the classical Cauchy rigidity theorem, the proof for showing two convex bounded face-congruent *c*-polyhedra are globally congruent involves a combinatorial lemma (exactly the same as that used in Cauchy's rigidity theorem) and a geometric lemma relying on an arm lemma. Using bounded polyhedra ensures the arm lemma works correctly.

4.3.2 Rigidity of Convex Circle-Polyhedra

Theorem 4.3.10 (Bowers, Bowers and Pratt)**.** *Any two convex and proper non-unitary c-polyhedra with Möbius-congruent faces that are based on the same oriented abstract spherical polyhedron and are consistently oriented are Möbius-congruent.*

4.4 Rigidity of Inversive Distance Circle Packings

Inversive distance circle packings are special cases of *c*-polyhedra, so Theorem 4.3.10 can be translated using the terminology outlined below. Here, we establish the state of the art for rigidity of inversive distance circle packings, after which we show how inversive distance circle packings can be modified to guarantee rigidity.

Definition 4.4.1. *A* ***circle packing*** *for an oriented, edge-labeled triangulation K_Γ of $\mathbb{S}^2$, with edge label $\Gamma : E(K) \to [0, \pi/2]$, is a collection $\mathcal{C} = \{C_v : v \in V(K_\Gamma)\}$ of circles in $\mathbb{S}^2$ centered at the vertices of the triangulation so that the two circles C_v and C_w meet at angle $\Gamma(e)$ whenever $e = vw$ is an edge of K.*

Definition 4.4.2. *Let $\mathcal{K}$ be an oriented triangulation of $\mathbb{S}^2$. A* ***unitary circle packing*** *P of the sphere is a collection of circles $\mathcal{C} = \{C_v : v \in V(\mathcal{K})\}$ centered at each vertex of $V(\mathcal{K})$ respectively, such that for each edge $uv \in E(\mathcal{K})$, circles C_u and C_v are tangent. The underlying edges of the circle packing are isomorphic to geodesics of the sphere.*

Definition 4.4.3. *An* ***inversive distance circle packing*** *for an edge-labeled triangulation K_Γ of $\mathbb{S}^2$ is a collection $\mathcal{C} = \{C_v : v \in V(K)\}$ of circles in $\mathbb{S}^2$ with four properties:*

(i) $\mathcal{C}$ is a circle realization for K_Γ;

(ii) when uvw is a face of K, the centers of C_u, C_v, and C_w do not lie on a great circle.

(iii) joining all the pairs of centers of adjacent circles C_u and C_v by geodesic segments of $\mathbb{S}^2$ produces a triangulation of $\mathbb{S}^2$, necessarily isomorphic with K.

We now formally write out the complete statement of the celebrated Koebe-Andre'ev Thurston Thoerem.

Theorem 4.4.4 (KAT Theorem for the Riemann sphere.)**.** *Let K be an oriented simplicial triangulation of $\mathbb{S}^2$, different from the tetrahedral triangulation, and let $\Gamma : E(K) \to [0, \pi/2]$ be a map assigning angle values to each edge of K. Assume that the following two conditions hold.*

(i) If e_1, e_2, e_3 for a closed loop of edges from K with $\Sigma_{i=1}^{3}\Gamma(e_i) \geq \pi$, then e_1, e_2, and e_3 form the boundary of a face of K.

(ii) If e_1, e_2, e_3, e_4 form a closed loop of edges from K with $\Sigma_{i=1}^{4}\Gamma(e_i) = 2\pi$, then e_1, e_2, e_3, and e_4 form the boundary of the union of two adjacent faces of K.

Then there is a realization of K as a geodesic triangulation of $\mathbb{S}^2$ and a family $\mathcal{C} = \{C_v : v \in V(K)\}$ of circles centered at the vertices of the triangulation so that the two circles C_v and C_w meet at angle $\Gamma(e)$ whenever $e = vw$ is an edge of K. The circle packing $\mathcal{C}$ is unique up to Möbius transformations.

Circle packings where edge labels are between 0 and $\pi/2$ are completely characterized by the Koebe-Andre'ev-Thurston Theorem abobe. As soon as the edge-label requirements are relaxed, inversive distance circle packings with the conditions above are no longer guaranteed to be rigid. Adding in the notion of convexity resolves the issue.

Theorem 4.4.5 (Bowers, Bowers and Pratt)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-unitary, inversive distance circle packings with ortho-circles for the same oriented edge-labeled triangulation of the 2-sphere $\mathbb{S}^2$. If $\mathcal{C}$ and $\mathcal{C}'$ are convex and proper, then there is a Möbius transformation T such that $T(\mathcal{C}) = \mathcal{C}'$.*

This is an example of adding a qualitative condition instead of more quantitative conditions on the configuration of circles. Of course, the work of [13] says that if all inversive distance information is known, then two collections are inversive-congruent, but, as was the case for Theorem 4.2.13, not all inversive distance information between circles is necessary for rigidity. We now go through the work of showing a sufficient amount of extra inversive distance information needed for Möbius-congruence between two general inversive distance circle packings.

We begin with developing an analogous notion of consistent orientation between configurations of circles which are not, in general, convex. The following definition is set up for inversive distance circle packings of either $\mathbb{S}^2$ or $\mathbb{D}$.

Definition 4.4.6. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two inversive distance circle packings based on the same oriented, edge-labeled triangulation K. Let f be a face of K. Then $\mathcal{C}$ and $\mathcal{C}'$ are said to* ***coincide with respect to*** *f if corresponding ortho-circles O_f and O'_f in $\mathcal{C}$ and $\mathcal{C}'$ can be oriented so that every oriented circle C in $\mathcal{C}$ is segregated from O_f if and only if C' in $\mathcal{C}'$ is segregated from O'_f. $\mathcal{C}$ and $\mathcal{C}'$* ***coincide*** *provided they coincide with respect to every face of K. Again, we additionally require that no c-face is degenerate by assuming that the circles corresponding to the vertices of a face in K are not coaxial.*

Lemma 4.4.7. *Let $\mathcal{C}$ and $\mathcal{C}'$ be inversive distance circle packings based on the same oriented, edge-labeled triangulation K, where $\mathcal{C}$ and $\mathcal{C}'$ coincide. Then either:*

(i) For every oriented face $f = u_1u_2u_3$ of K, the circles C_1, C_2, C_3 in $\mathcal{C}$ and C'_1, C'_2, C'_3 in $\mathcal{C}'$ are met in the same order as one progresses around O_f^+ and $O_f^{+\prime}$ respectively, in the direction of each ortho-circle's orientation, starting at C_1 and C'_1, or

(ii) For every oriented face $f = u_1u_2u_3$ of K, the circles C_1, C_2, C_3 in $\mathcal{C}$ and C'_1, C'_2, C'_3 in $\mathcal{C}'$ are met in opposite order as one progresses around O_f^+ and $O_f^{+\prime}$ respectively, in the direction of each ortho-circle's orientation, starting at C_1 and C'_1.

As in Lemma 4.3.6, if *(ii)* occurs, apply the antipodal map and change the orientation of all the circles in one of the inversive distance circle packings to match the other. In this way, we will assume *(i)* always occurs when $\mathcal{C}$ and $\mathcal{C}'$ coincide, and in this case we say two such coincident inversive distance circle packings are ***oriented consistently***.

There is a distinction made here between two inversive distance circle packings being oriented consistently, and an inversive distance circle packing being oriented consistently with a triangulation K. In the latter, while $\mathcal{C}$ and $\mathcal{C}'$ may be oriented consistently with one another, a face f of K may be consistently oriented with O_f^+ and $O_f^{+\prime}$ in $\mathcal{C}$ and $\mathcal{C}'$ respectively, or f may be oppositely oriented to both O_f^+ and $O_f^{+\prime}$.

Observe that because of the assumption of (i) in Lemma 4.4.7 that two inversive distance circle packings are oriented consistently then any two adjacent faces $f = u_1u_2u_3$ and $g = u_3u_2u_4$ of the underlying triangulation K, sharing unoriented edge $e = u_2u_3$, yield collections of circles $\{C_1, C_2, C_3, C_4\}$ and $\{C_1', C_2', C_3', C_4'\}$, with their ortho-circles, that satisfy Definition 4.2.8.

Lemma 4.4.8. *Let K be a triangulation of the closed disk $\mathbb{D}$. There exists a vertex on the boundary of $\mathbb{D}$ that is adjacent to exactly two other vertices in the boundary of $\mathbb{D}$.*

Proof. Let K be a triangulation of the closed disk $\mathbb{D}$, with n vertices on the boundary of $\mathbb{D}$. The boundary vertices of K form a cycle in the 1-skeleton of K; label the boundary vertices $v_1, \dots, v_n$, in a counter clockwise direction about this cycle, where v_n is adjacent to v_1. Consider the subgraph G of the 1-skeleton of K with vertex set $\{v_1, \dots, v_n\}$, the boundary vertices of K, and edge set composed of all the edges in K incident only to boundary vertices of K. Since $v_1v_2 \dots v_nv_1$ is a cycle, each vertex v_j is adjacent to at least two other vertices. If every vertex is degree 2, we are done, so assume there is at least one vertex adjacent to a vertex other than its adjacencies in the cycle $v_1v_2 \dots v_nv_1$. Travel the cycle in a counter clockwise direction starting from v_1, and find an outermost cycle, that is, a cycle which does not contain any other cycles. Label the outermost cycle $v_iv_{i+1} \dots v_{k-1}v_kv_i$. Then within this outermost cycle, vertex $v_j \in \{v_{i+1}, v_{i+2}, \dots, v_{k-2}, v_{k-1}\}$ is not adjacent to any vertices except v_{j-1} and v_{j+1}. Hence, v_j is a degree 2 vertex in G, and so is adjacent to exactly 2 vertices on the boundary of D in K. □

We use this lemma to make the following observation. Let K be an oriented triangulation of a closed disk $\mathbb{D}$, and let v be a vertex in the boundary of D which is adjacent to exactly two other vertices in the boundary. Then the oriented subgraph of K excluding the star of v is also an oriented triangulation of $\mathbb{D}$. The main theorems use induction on the number of vertices in a triangulation, so this observation will be used frequently.

In this section, we continue to work under the assumption that all c-faces are non-degenerate, and in particular, are hyperbolic. The key difference here is that the inversive distance circle packings are not convex. However, we keep the condition that $\mathcal{C}$ and $\mathcal{C}'$ realizing the same oriented triangulation K must be consistently oriented with K. Circles belonging to $\mathcal{C}$ or $\mathcal{C}'$ not in a c-face generated by an orthocircle O may take on any inversive distance with O now.

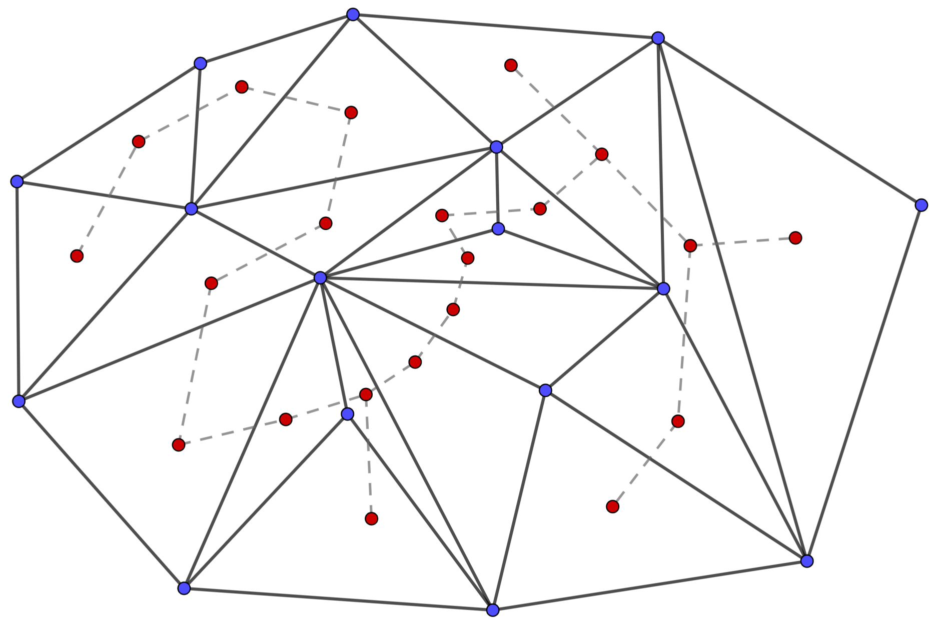

Figure 4.7: An example of a triangulation K of closed unit disk $\mathbb{D}$ (in blue), and a face-spanning tree T (in red) of K^*, where K^* is the dual graph of K.

Definition 4.4.9. *Let K be an oriented triangulation of $\mathbb{D}$. Consider the dual graph K^* of K, where a face f in K is represented by a vertex v_f in K^*, and where there is an edge between v_{f_i} and v_{f_j} whenever f_i and f_j are adjacent faces in K. Call a spanning tree of K^* a* ***face-spanning tree*** *T.*

Let $\mathcal{C}$ be a circle packing based on K. Let f_i and f_j be adjacent faces sharing vertices u and w, where $f_i = uvw_i$ and $f_j = uvw_j$. When an edge e_{ij} of T between vertices v_{f_i} and v_{f_j} in dual graph K^* is equipped with an edge label $\beta : E(T) \to \mathbb{R}$, then $\beta(e_{ij}) = (C_{w_i}, C_{w_j})$, where C_{w_i} and C_{w_j} are circles in $\mathcal{C}$ corresponding to vertices w_i and w_j respectively.

Theorem 4.4.10. *Let K be an edge-labeled, oriented triangulation of $\mathbb{D}$. Let $\mathcal{C}$ and $\mathcal{C}'$ be two inversive distance circle packings of $\mathbb{D}$ based on triangulation K, where $\mathcal{C}$ and $\mathcal{C}'$ are coincident and oriented consistently. Let T be an edge-labeled face spanning tree of K^*. There is a subgraph G of T such that if $\mathcal{C}$ and $\mathcal{C}'$ realize G, then there is a Möbius transformation ϕ such that $\phi(\mathcal{C}) = \mathcal{C}'$.*

Proof. We proceed by induction on the number of vertices in the triangulation K of $\mathbb{D}$.

For an edge-labeled, oriented triangulation K of $\mathbb{D}$ with $n = 3$ vertices, two inversive distance circle packings of $\mathbb{D}$, labeled $\mathcal{C}_3$ and $\mathcal{C}'_3$, are independent collections of 3 circles in $\mathbb{S}^2$ with supporting ortho-circles and corresponding equal inversive distances, so there is an inversive transformation ϕ such that $\phi(\mathcal{C}) = \mathcal{C}'$. Since orientations are consistent between $\mathcal{C}$ and $\mathcal{C}'$, ϕ is a Möbius transformation.

Assume for any edge-labeled, oriented triangulation K_n of $\mathbb{D}$, and any two inversive distance circle packings $\mathcal{C}_n$ and $\mathcal{C}'_n$ realizing K_n that coincide and are oriented conistently, there is a subgraph of a face-spanning tree of K_n^* such that if $\mathcal{C}_n$ and $\mathcal{C}'_n$ realize the subgraph, then $\mathcal{C}_n$ and $\mathcal{C}'_n$ are Möbius-equivalent. Let K_{n+1} be an edge-labeled, oriented triangulation of $\mathbb{D}$ with $n+1$ vertices, where $n+1 \geq 4$. Let $\mathcal{C}_{n+1}$ and $\mathcal{C}'_{n+1}$ be two consistently-oriented inversive distance circle packings realizing K_{n+1}. For vertex v on the boundary of $\mathbb{D}$, adjacent to no more than two other vertices on the boundary of $\mathbb{D}$, let $K_{(n+1),v}$ be the oriented, edge-labeled triangulation excluding vertex v and its incident edges. The subcollections $\mathcal{C}_{(n+1),v}$ and $\mathcal{C}'_{(n+1),v}$, excluding circle C_v and C'_v in $\mathcal{C}_{n+1}$ and $\mathcal{C}'_{n+1}$ respectively, are each consistently-oriented inversive distance circle packings realizing $K_{(n+1),v}$ which coincide, so by the inductive hypothesis, there is a subgraph $G_{(n+1),v}$ of the face-spanning tree $T_{(n+1),v}$ such that if $\mathcal{C}_{(n+1),v}$ and $\mathcal{C}'_{(n+1),v}$ realize $G_{(n+1),v}$, then there is a Möbius transformation ϕ such that $\phi(\mathcal{C}_{(n+1),v}) = \mathcal{C}'_{(n+1),v}$. We consider two cases to determine where ϕ sends C_v.

Case 1: Vertex v is adjacent to three or more vertices in K_{n+1} that are not coaxial. In this case, $G_{n+1} = G_{(n+1),v}$. Call the vertices in K_{n+1} adjacent to v vertices $u_1, \ldots, u_k$, where $k \geq 3$. The vertices $u_2, \ldots, u_{k-1}$ are interior vertices, and u_1 and u_k are the two boundary vertices adjacent to v. Label the corresponding oriented circles in $\mathcal{C}_{n+1}$ and $\mathcal{C}'_{n+1}$ as C_i and C'_i for each $i = 1, \ldots, k$. Without loss of generality, assume that C_1, C_2, C_3 are not coaxial. If $\mathcal{C}_{n+1}$ and $\mathcal{C}'_{n+1}$ realize graph $G_{n+1} = G_{(n+1),v}$, then for $i = 1, 2, 3$, $(\phi(C_v), C'_i) = (\phi(C_v), \phi(C_i)) = (C_v, C_i) = (C'_v, C'_i)$. Since C_1, C_2, and C_3 are not coaxial, $\{C_1, C_2, C_3\}$ is an independent collection of circles. Since $\mathcal{C}_{n+1}$ and $\mathcal{C}'_{n+1}$ are coincide and oriented consistently, and $\{C'_1, C'_2, C'_3\}$ is an independent collection of 3 circles in $\mathbb{S}^2$, by Lemma 4.2.11, there is a unique Möbius transformation σ such that $\sigma(C_v) = C'_v$ and $\sigma(C_i) = C'_i$ for $i = 1, 2, 3$. Since $\phi(C_i) = \sigma(C_i)$ for $i = 1, 2, 3$ by Corollary 4.1.21, $\phi = \sigma$, so $\phi(C_v) = C'_v$.

Case 2: Vertex v is adjacent to exactly two vertices in K_{n+1}, or all vertices which are coaxial. Call the vertices in K_{n+1} adjacent to v vertices $u_1, \ldots, u_k$, where $k \geq 2$. Label the corresponding

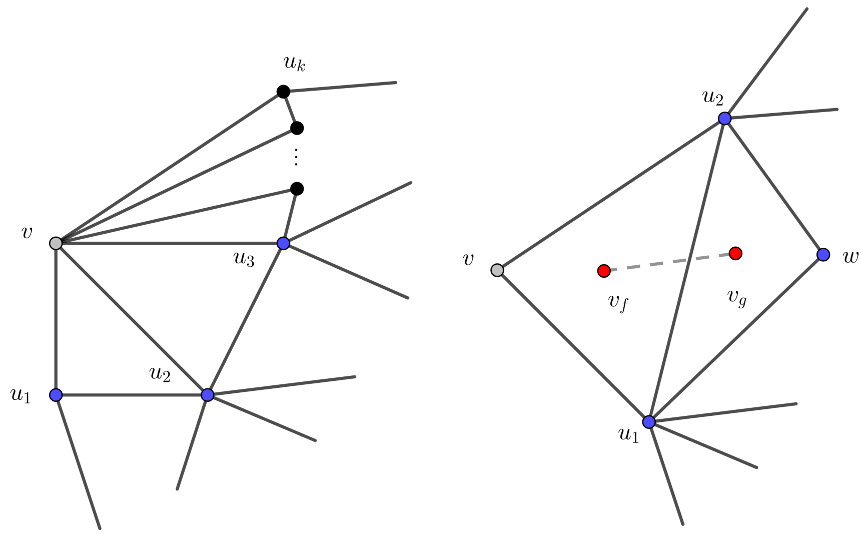

Figure 4.8: Case 1 (left) and Case 2 (right). The vertices in K_{n+1} used to uniquely place vertex v in each case are shown in blue; in Case 2, the extra edge in T_{n+1} needed to uniquely place v is shown between the red vertices.

oriented circles in $\mathcal{C}_{n+1}$ and $\mathcal{C}'_{n+1}$ as C_i and C'_i for $i = 1, \ldots k$. Call the vertex opposite v, over edge $e_{12} = u_1u_2$, vertex w with corresponding circle C_w and C'_w respectively. Label face $f = u_1vu_2$ and $g = u_1u_2w$. Then in the dual graph K^*_{n+1} of K_{n+1}, consider the vertices v_f and v_g corresponding to faces f and g. Let $e_{fg} = v_fv_g$ be the edge between v_f and v_g. Call graph G_{n+1} the subgraph of face-spanning tree T_{n+1} gotten from adding a labeled edge e_{fg} to $G_{(n+1),v}$ in K^*_{n+1}. If $\mathcal{C}_{n+1}$ and $\mathcal{C}'_{n+1}$ realize G_{n+1}, then $(\phi(C_v), C'_i) = (C_v, C_i) = (C'_v, C'_i)$ for each $i = 1, 2$ and $(\phi(C_v), C'_w) = (C'_v, C'_w)$. Since $\{C_w, C_1, C_2\}$ is an independent collection of 3 circles in $\mathbb{S}^2$ and $\mathcal{C}_{n+1}$ and $\mathcal{C}'_{n+1}$ coincide and are oriented consistently, by Lemma 4.2.11, there is a unique Möbius transformation σ such that $\sigma(C_v) = C'_v$, $\sigma(C_w) = C'_w$, and $\sigma(C_i) = C'_i$ for $i = 1, 2$. Since $\phi(C_w) = \sigma(C_w)$ and $\phi(C_i) = \sigma(C_i)$ for $i = 1, 2$, by Corollary 4.1.21, $\phi = \sigma$, so $\phi(C_v) = C'_v$. □

Note in Case 1 that only edges in $G_{(n+1),v^*} = G_{n+1}$ are used in addition to the triangulation of K_{n+1} to say the circle packings are Möbius-equivalent. In case 2, a new edge is added in G_{n+1}. This presents an algorithm for constructing a subgraph of a face-spanning tree sufficient for making a collection of circles rigid, where, in general, not all of a face-spanning tree need be used. Of course, this can no longer be called an inversive distance circle packing, because the underlying structure of known inversive distances is no longer a triangulation pattern.

Theorem 4.4.11. *Let K be an edge-labeled, oriented triangulation of $\mathbb{S}^2$. Let $\mathcal{C}$ and $\mathcal{C}'$ be two inversive distance circle packings of $\mathbb{S}^2$ based on triangulation K, where $\mathcal{C}$ and $\mathcal{C}'$ are coincident and consistently-oriented. Let T be a face spanning tree of K^*. There is an edge-labeled subgraph G of T such that if $\mathcal{C}$ and $\mathcal{C}'$ both realize the same edge-labeling on G, then there is a Möbius transformation ϕ such that $\phi(\mathcal{C}) = \mathcal{C}'$.*

Proof. Let K be an edge-labeled, oriented triangulation of $\mathbb{S}^2$, and let $\mathcal{C}$ and $\mathcal{C}'$ be two inversive distance circle packings of $\mathbb{S}^2$ realizing K that coincide and are oriented consistently. Let v be any vertex of K. Note that v must be adjacent to at least three vertices. Call these vertices $u_1, \ldots, u_k$, and call corresponding circles in $\mathcal{C}$ and $\mathcal{C}'$, respectively, C_i and C_i', for $i = 1, \ldots, k$. There must be three vertices in $u_1, \ldots, u_k$ such that the corresponding circles in $\mathcal{C}$ and $\mathcal{C}$ are not coaxial; otherwise, there is a c-face which is degenerate in either $\mathcal{C}$ or $\mathcal{C}'$. Without loss of generality, assume the circles corresponding to $i = 1, 2, 3$ are not coaxial, so that collections $\{C_1, C_2, C_3\}$ and $\{C_1', C_2', C_3'\}$ are independent. Let K_v be a triangulation of $\mathbb{D}$ that excludes v and its incident edges, with corresponding inversive distance circle packings of $\mathcal{C}_v$ and $\mathcal{C}_v'$ of $\mathbb{D}$ which exclude circle C_v and C_v' respectively corresponding to vertex v in K. Let T_v be a face-spanning tree of K_v^*. Then by Theorem 38, there is a subgraph G_v of T_v such that if $\mathcal{C}_v$ and $\mathcal{C}_v'$ realize the same edge-labeling on G_v, then there is a Möbius transformation ϕ where $\phi(\mathcal{C}_v) = \mathcal{C}_v'$. Furthermore, if this is the case, then $\phi(C_v) = C_v'$ because $(\phi(C_v), C_i') = (C_v, C_i) = (C_v', C_i')$ for $i = 1, 2, 3$, and $\mathcal{C}$ and $\mathcal{C}'$ coincide and are oriented consistently. □

BIBLIOGRAPHY

[1] Xiliang Bao, Francis Bonahon. *Hyperideal polyhedra in hyperbolic 3-space*. Bulletin de la Société Mathématique de France, Volume 130 (2002) no. 3, pp. 457-491. doi : 10.24033/bsmf.2426.

[2] A.F. Beardon. *The Geometry of Discrete Groups*. Springer-Verlag, New York (1983).

[3] A. F. Beardon, D. Minda. *Conformal automorphisms of finitely connected regions*. In P. Rippon & G. Stallard (Eds.), *Transcendental Dynamics and Complex Analysis*. (London Mathematical Society Lecture Note Series, pp. 37-73). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511735233.004

[4] Philip L. Bowers. *Combinatorics encoding geometry: the legacy of Bill Thurston in the story of one theorem*. EMS, to appear

[5] John C. Bowers, Philip L. Bowers, Kevin Pratt. *Rigidity of circle polyhedra in the 2-sphere and of hyperideal polyhedra in hyperbolic 3-space*. Trans. Amer. Math. Soc., 371 (2019), 4215-4249, https://doi.org/10.1090/tran/7483

[6] John C. Bowers, Philip L. Bowers. *Ma-Schlenker c-Octahedra in the 2-Sphere*. Discrete Comput. Geom., 60, 9–26 (2018). https://doi.org/10.1007/s00454-017-9928-1

[7] Philip L. Bowers, Monica K. Hurdal. *Planar Conformal Mappings of piecewise flat surfaces* Hege HC., Polthier K. (eds) Visualization and Mathematics III. Mathematics and Visualization. Springer, Berlin, Heidelberg, (2003).

[8] Philip L. Bowers, Kenneth Stephenson. *Conformal Tilings I: Foundations, Theory, and Practice*. Conform. Geom. Dyn., 21, 1-63, (2017). https://doi.org/10.1090/ecgd/304

[9] Philip L. Bowers, Kenneth Stephenson. *Conformal Tilings II: Local Isomorphism, Hierarchy, and Conformal Type*. Conform. Geom. Dyn. 23, 52-104, (2019). https://doi.org/10.1090/ecgd/333

[10] James W. Cannon. *The combinatorial structure of cocompact discrete hyperbolic groups*. Geometriae Dedicata, 16, 123-148, (1984).

[11] J.W. Cannon, W.J. Floyd, W.R. Parry. *Finite Subdivision Rules*. Conform. Geom. Dyn., 5, 153-196, (2001). https://doi.org/10.1090/S1088-4173-01-00055-8

[12] John Conway. *Functions of a Complex Variable II*. Springer-Verlag, New York (1995).

[13] Edward Crane, Ian Short. *Rigidity of configurations of balls and points in the n-sphere.* The Quarterly Journal of Mathematics,Volume 62, Issue 2, (2011), Pages 351–362, https://doi.org/10.1093/qmath/hap044

[14] Paul Koebe. *Uber die Uniformisierung beliebiger analytischer Kurven, III.* Nach. Ges. Wiss. Gott. (1908), no. 337-358.

[15] Paul Koebe. *Abhandlungen zur Theorie der konformen Abbildung: VI. abbildung mehrfach zusammen-hängender Bereiche auf Kreisbereiche, etc.* Math. Z. **7**(1920), 235-301

[16] Paul Koebe. *Kontaktprobleme der konformen Abbidung.* Ber. Säcs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **88** (1936), 141-164.

[17] Igor Pak. *Lectures on Discrete and Polyhedral Geometry.* 25-27, (2010). https://www.math.ucla.edu/ pak/geompol8.pdf

[18] William P. Thurston. *The geometry and topology of* 3*-manifolds.* Lecture Notes: Princeton University, (1980).

[19] John G. Ratcliffe. *Foundations of Hyperbolic Manifolds.* Springer-Verlag New York, (2006).

[20] Igor Rivin. *A Characterization of Ideal Polyhedra in Hyperbolic 3-Space.* Annals of Mathematics, 143, 51-70, (1996).

[21] Roland K.W. Roeder, John H. Hubbard, William D. Dunbar. *Andreev's Theorem on hyperbolic polyhedra.* Annales de l'Institut Fourier, 57, 825-882, (2007).

BIOGRAPHICAL SKETCH

Opal Jane Graham was born December 16, 1991 in Forth Worth, Texas. She received her Associates of Arts from Florida State College at Jacksonville, and finished her Bachelor of Arts majoring in Mathematics at University of North Florida in Jacksonville, Florida during Summer 2014. That August, she started the Graduate program in the Department of Mathematics at Florida State University. She received her Masters of Science in August 2016, and earned her PhD under the direction of Dr. Philip Bowers.